FunSP, EXAM OF JANUARY 17, 2023

NOTE: These solution topics are more detailed than strictly required, they serve didactic purposes

1. a) C − 2 This is the easiest association given that the two poles on the real axis near $z=1$ and $z=-1$ create peaks in the frequency response magnitude for $\omega=0$ rad. and $\omega=\pi$ rad., respectively.

A − 1 The pairs of zeros on the real axis near $z=1$ and $z=-1$ give rise to valleys in the frequency response magnitude for $\omega=0$ rad. and $\omega=\pi$ rad., respectively.

B − 3 The zeros on the real axis near $z=-1$ and $z=1$, and on the imaginary axis near $z=j$ and $z=-j$ produce a similar impact on the frequency response magnitude given that their distance to the origin of the z plane is either 0.8 or 1/0.8 and no other poles or zeros exist nearly. As such, as expected, the magnitude of the frequency response shows vallys for all multiples of $\pi/2$ rad.

b) Only system A has a linear-phase response because it is FIR (all poles are located at the origin of the z plane) and all zeros, being real-valued, exist as reciprocal pairs. Other possibilities would exist if zeros are arranged as reciprocal-conjugate pairs. That is the case of the BC cascade. In fact, in this case, two poles cancel two zeros and two pairs of reciprocal-conjugate zeros exist: $0.8\, e^{j\pi/2}$, $1/0.8\; e^{j\pi/2}$ and $0.8\, e^{-j\pi/2}$, $1/0.8\; e^{-j\pi/2}$.

That is also the case of the ABC cascade given that it combines the two pairs of reciprocal zeros due to system A, and the two pairs of reciprocal-conjugate zeros due to the BC cascade.

c) In answering this question it is important to recognize the meaning of $H(z)$ being transformed into $H(-z)$. $H(-z)$ means $H(\frac{z}{\alpha})$ where $\alpha = e^{\pm j\pi}$, which reflects the complex modulation property of the Z Transform whose effect is, in this case, to rotate the Z plane left, or right, by π rad., which is equivalent to delay, or advance, the frequency response magnitude by π rad. As this is true for all systems illustrated (A, B, C), the sentence is true.

$\boxed{2.}$ a) The system has two poles: r and $-r$, and has two zeros: jr and $-jr$, with $r = 0.8$.

Thus, we may write:

$$H(z) = \frac{(z-jr)(z+jr)}{(z-r)(z+r)} = \frac{z^2+r^2}{z^2-r^2}$$

and, considering that the system is causal, all powers of z must be either zero or negative and, thus,

$$H(z) = \frac{1 + r^2 z^{-2}}{1 - r^2 z^{-2}} = \frac{1 + (r z^{-1})^2}{1 - (r z^{-1})^2}, \quad |z| > r$$

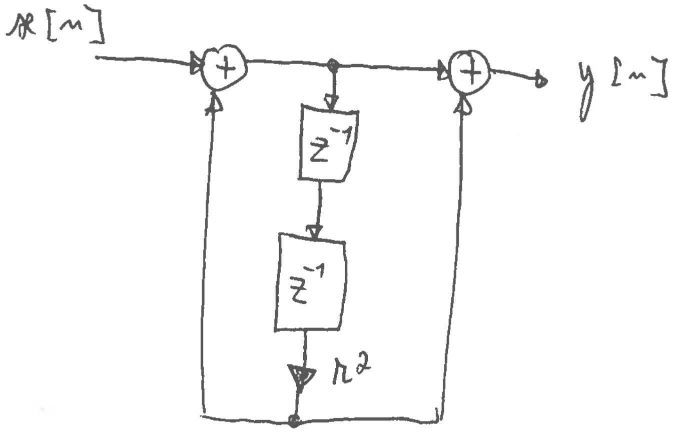
Since $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + r^2 z^{-2}}{1 - r^2 z^{-2}}$, reducing to the same denominator we obtain (on the assumption that $X(z) \neq 0$ and $r^2 z^{-2} \neq 1$):

$$Y(z)(1 - r^2 z^{-2}) = (1 + r^2 z^{-2}) X(z)$$

which, taking advantage of the Z-Transform properties and inverse Z-Transform, leads to:

$$y[n] = x[n] + r^2 x[n-2] + r^2 y[n-2]$$

A possible canonic realization structure is:



b) $H(z) = \frac{1 + r^2 z^{-2}}{1 - r^2 z^{-2}}, \quad |z| > r$

In the Fourier domain we have:

$$H(e^{jw}) = \frac{1 + r^2 e^{-j2w}}{1 - r^2 e^{-j2w}}$$

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and the magnitude is given by

$$|H(e^{j\omega})| = \sqrt{H(e^{j\omega})H^*(e^{j\omega})} = \sqrt{\frac{(1+r^2e^{-j2\omega})(1+r^2e^{j2\omega})}{(1-r^2e^{-j2\omega})(1-r^2e^{j2\omega})}}$$

$$= \sqrt{\frac{1+2r^2\cos 2\omega + r^4}{1-2r^2\cos 2\omega + r^4}}$$

which allows to easily obtain the maximum and minimum values:

$$\max_{\omega} |H(e^{j\omega})| = \sqrt{\frac{1+2r^2+r^4}{1-2r^2+r^4}} = \sqrt{\frac{(1+r^2)^2}{(1-r^2)^2}} = \left|\frac{1+r^2}{1-r^2}\right|$$

$$\min_{\omega} |H(e^{j\omega})| = \sqrt{\frac{1-2r^2+r^4}{1+2r^2+r^4}} = \sqrt{\frac{(1-r^2)^2}{(1+r^2)^2}} = \left|\frac{1-r^2}{1+r^2}\right|$$

e) $F_S = 200\,Hz$

$x_c(t) = 1 + \sin 300\pi t + 2\sin 500\pi t$

Ideal sampling leads to:

$$x[n] = x_c(t)\Big|_{t=nT=\frac{n}{F_S}} = 1 + \sin\frac{300\pi n}{200} + 2\sin\frac{500\pi n}{200}$$

$$= 1 + \sin n\frac{3\pi}{2} + 2\sin n\frac{5\pi}{2}$$

$\omega_0 = 0$ rad. $\quad \omega_1 = \frac{3\pi}{2}$ rad. $\quad \omega_2 = \frac{5\pi}{2}$

Thus, in the Nyquist range:

$$\omega_1 = \frac{3\pi}{2} + K2\pi = \frac{3\pi + K4\pi}{2}\Bigg|_{K=-1} = -\frac{\pi}{2} \text{ rad.}$$

$$\omega_2 = \frac{5\pi}{2} + K2\pi = \frac{5\pi + K4\pi}{2}\Bigg|_{K=-1} = \frac{\pi}{2} \text{ rad.}$$

which leads to $x[n] = 1 + \sin(-n\frac{\pi}{2}) + 2\sin(n\frac{\pi}{2})$

$$= 1 - \sin(n\tfrac{\pi}{2}) + 2\sin(n\tfrac{\pi}{2}) = 1 + \sin(n\tfrac{\pi}{2})$$

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d) Under ideal reconstruction conditions and given that $H(e^{j\omega})$ is conjugate-symmetric, i.e. $H(e^{j\omega}) = H^*(e^{-j\omega})$, which is the same as stating that $h[n]$ is real-valued,

then: $$y[n] = y_e(t)\Big|_{t=nT=\frac{n}{F_S}}$$

and: $$y[n] = 1 \times H(e^{j\omega})\Big|_{\omega=0} + \left|H(e^{j\frac{\pi}{2}})\right| \sin\left(n\frac{\pi}{2} + \angle H(e^{j\frac{\pi}{2}})\right)$$

which requires us to find:

$$H(e^{j\omega})\Big|_{\omega=0} = \frac{1+r^2}{1-r^2}$$

and

$$\left|H(e^{j\frac{\pi}{2}})\right| = \left|\frac{1+r^2 e^{-j\pi}}{1-r^2 e^{-j\pi}}\right| = \left|\frac{1-r^2}{1+r^2}\right|$$

Finally, $$y[n] = y_e(t)\Big|_{t=n/F_S} = \frac{1+r^2}{1-r^2} + \frac{|1-r^2|}{1+r^2} \sin\left(\frac{n}{F_S}\frac{F_S\pi}{2} + \angle H(e^{j\frac{\pi}{2}})\right)$$

$$= \frac{1+r^2}{1-r^2} + \frac{|1-r^2|}{1+r^2} \sin\left(100\pi t + \angle H(e^{j\frac{\pi}{2}})\right)\Big|_{t=\frac{n}{F_S}}$$

which means that the output sinusoidal frequencies are 0 Hz and 50 Hz, and their magnitudes are $\frac{1+r^2}{|1-r^2|}$ and $\frac{|1-r^2|}{1+r^2}$, respectively.

$\boxed{3.}$

a) This code takes as input the most recent A/D sample, $w0 = x[n]$ and, depending on the value of the "myfilter" variable, computes:

in case myfilter == FIR

$$y[n] = x[n] + \alpha\, w[n-N]$$

and, due to the fact that the delay chain $w[n]$ stores

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the input signal, i.e. $w[n] = x[n]$, then the above difference equation is the same as

$$y[n] = x[n] + \alpha x[n-N]$$

in case myfilter == IIR two difference equations apply:

$$\begin{cases} w[n] = x[n] + \beta w[n-N] \\ y[n] = w[n] \end{cases}$$

In the Z-domain these correspond to:

$$\begin{cases} W(z) = X(z) + \beta z^{-N} W(z) \\ Y(z) = W(z) \end{cases} \Leftrightarrow \begin{cases} W(z) = \dfrac{X(z)}{1-\beta z^{-N}} \\ Y(z) = W(z) \end{cases} \Leftrightarrow \begin{cases} - \\ Y(z) = \dfrac{X(z)}{1-\beta z^{-N}}, \quad |z| > \sqrt[N]{\beta} \end{cases}$$

which means, returning back to the discrete-time domain:

$$y[n] = x[n] + \beta y[n-N]$$

b) In this case, the new difference equations become:

$$\begin{cases} w[n] = x[n] + \alpha w[n-N] \\ y[n] = w[n-N] - \alpha w[n] \end{cases}$$

In the Z-domain:

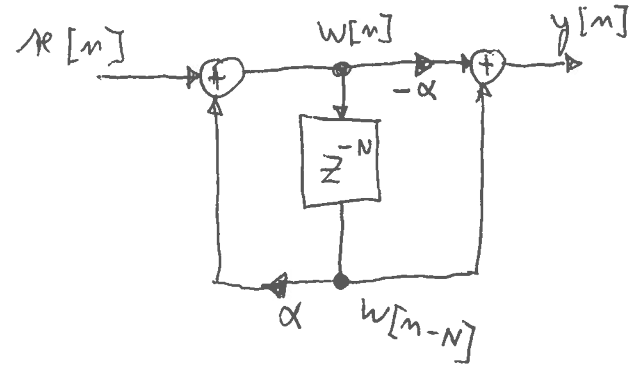
$$\begin{cases} W(z) = X(z) + \alpha z^{-N} W(z) \\ Y(z) = z^{-N} W(z) - \alpha W(z) \end{cases} \Leftrightarrow \begin{cases} W(z) = \dfrac{X(z)}{1-\alpha z^{-N}}, \quad |z| > \sqrt[N]{\alpha} \\ Y(z) = (z^{-N} - \alpha) W(z) \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ Y(z) = \dfrac{-\alpha + z^{-N}}{1 - \alpha z^{-N}}, \quad |z| > \sqrt[N]{\alpha} \end{cases}$$

and the new difference equation results as:

$$y[n] = x[n-N] - \alpha x[n] + \alpha y[n-N]$$

The realization structure is



which an N^{th} order all-pass system.

4. a) Using the inverse DFT:

$$y[n] = \frac{1}{2N}\sum_{k=0}^{2N-1} Y[k]\, W_{2N}^{-km} = \frac{1}{2N}\sum_{k=0}^{N-1} Y[2k]\, W_{2N}^{-2km} + \frac{1}{2N}\sum_{k=0}^{N-1} \underbrace{Y[2k+1]}_{=0}\, W_{2N}^{-(2k+1)m}$$

$$= \frac{1}{2N}\sum_{k=0}^{N-1} \underbrace{Y[2k]}_{=2X[k]}\, W_{N}^{-km}$$

$$= \frac{1}{2N}\sum_{k=0}^{N-1} 2X[k]\, W_{N}^{-km} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_{N}^{-km} = x[m]$$

Given that $x[n]$ is N-periodic, and $y[n]$ is 2N-periodic, then:

$y[n] = x[(m)_N]$, $m = 0, 1, \ldots, 2N-1$

b) This code implements $Y[2k] = 2X[k]$, $k = 0, 1, \ldots, N-1$ and $Y[2k+1] = 0$, $k = 0, 1, \ldots, N-1$ and then computes $y[n] = IDFT\{Y[k]\}$ which delivers $y[n] = x[(m)_N]$

i.e. $y[n] \equiv [1\ 2\ 3\ 4\ 1\ 2\ 3\ 4]$

c) This code implements a circular right-shift of the $Y[K]$ spectrum by $\frac{L}{4}$, that is:

$$Z[K] = Y\left[\left(K-\frac{L}{4}\right)_L\right]$$

Now, using the DFT properties:

$$y[n] \xleftrightarrow{DFT} Y[K] \quad , \quad K=0,1,\dots,L-1$$

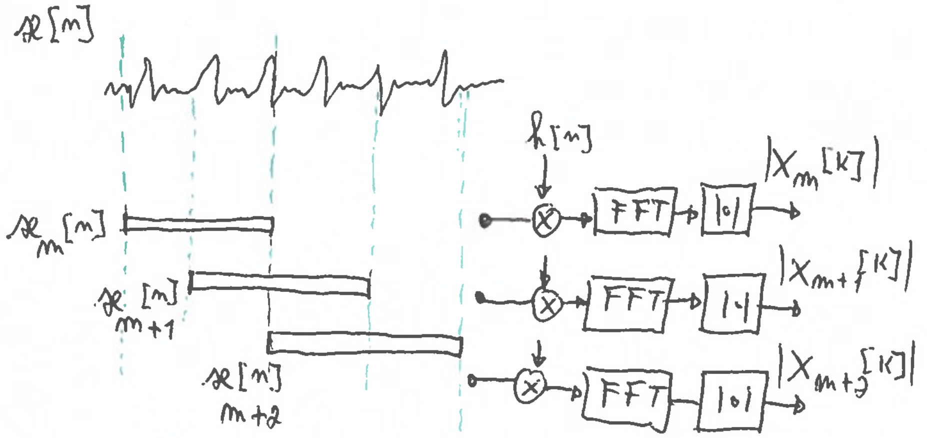
$$e^{j\frac{2\pi}{L}n\frac{L}{4}}\, y[n] \longleftrightarrow Y\left[\left(K-\frac{L}{4}\right)_L\right]$$

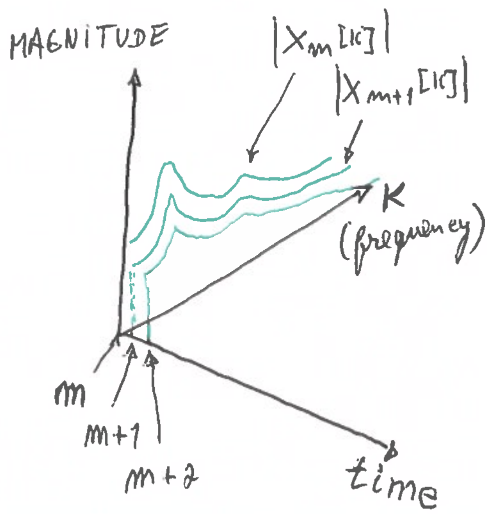
which means that the time-domain signal becomes $e^{jn\frac{\pi}{2}}\, y[n]$, $n=0,1,\dots,L-1$:

$$e^{jn\frac{\pi}{2}}\, y[n] = j^n\, y[n] \equiv \left[1 \;\; 2j \;\; -3 \;\; -4j \;\; 1 \;\; 2j \;\; -3 \;\; -4j\right]$$

$\boxed{5.}$

a) The sliding FFT analysis process can be schematically represented as follows (it was explained in a recent lecture):





The most important ideas are as follows:

- a signal $x[n]$ is segmented in segments of N samples with a given overlap between segments that are adjacent (but it can also be with no overlap);

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- each segment is multiplied by a time-domain window $h[n]$ (it can be the Rectangular Window, Hamming, Hanning, ...), the result is DFT/FFT transformed and, then, the magnitude is found;
- each magnitude spectrum is colored according to a color map such that different power spectral densities (PSD) are assigned different colors, or different gray levels;
- the collection of successive magnitude spectra are put side-by-side (i.e. they are abutted) giving rise to a 3D representation;
- when looked at from the top, the horizontal axis represents time, the vertical axis represents frequency, and the color or gray levels represent PSD which denote the 3rd axis that is perpendicular to the paper or to the screen. This 3D map represented as a 2D figure is called spectrogram. It informs about the spectral contents of a signal and how it evolves through time.

b) The FFT size determines the granularity, or resolution, of the DFT/FFT analysis; the larger the size, the finer (i.e. the better) the spectral resolution. It is therefore clear that diagrams B and D reflect the FFT size of 256.

This means that diagrams A and C reflect the FFT size of N. In order to find N, one should realize that the frequency axis in the diagrams represents the Nyquist range only, which is the same as saying that it represents a spectral analysis using $\frac{N}{2}$ sub-bands (the interpretation of the DFT/FFT as a filter bank has been explained in one of the lectures). By observing the A and C diagrams, one concludes that the granularity suggests $\frac{N}{2} = 16$, therefore $N = 32$.

In order to identify the window being used in each diagram, one has to consider the concept of "spectral leakage" that was studied in both lecture and LAB classes.

The rectangular window has the narrowest main lobe in its frequency response magnitude, which means that the width of the pass-band of each sub-band filter is the narrowest. The rectangular window has also the poorest main-to-side lobe attenuation in its frequency response magnitude, which means that the "far-end leakage" is very high, and, as a consequence, very significant levels of out-of-band energy (of the signal being analysed) are captured by most (or many) sub-bands. As a consequence, the spectrograms appear "blurred". This means that diagrams A and D reflect the usage of the Rectangular window. On the contrary, the Hanning window has a larger "near-end leakage" but a much less "far-end leakage" than the Rectangular window, which means that the bandwith of each pass-band is larger and, as a compensation, the isolation among sub-bands is better, i.e. levels of out-of-band energy are very low. As a consequence, signal representations in a spectrogram are thicker, but cleaner. This means that diagrams B and C reflect the usage of the Hanning window.

c) The signal includes:

- a narrow-band signal (pobably a sinusoid) whose center frequency increases linearly from 0 Hz and 8 KHz (the Nyquist frequency) between 0 sec. and 1.5 sec.;
- a constant-frequency sinusoid of 6 KHz that exists between 0.2 sec. and 0.8 sec.;
- a sinusoid that is modulated in frequency by another sinusoid and that exists between 0.8 sec. and 1.4 sec.; its center frequency is around 2 KHz, the frequency deviation is around 1 KHz, and the frequency modulation of the frequency-modulated sinusoid is $\frac{1}{1.4-0.8} \cong 1.67$ Hz

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