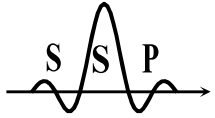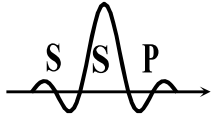# Summary (1/2)

- *Characterization and representation of discrete signals*
  - *Types of signals*
  - *Discrete-time signals*
    - *representation of a discrete-time signal*
    - *basic discrete-time signals*

- *Characterization and representation of discrete systems*
  - *Properties of discrete-time systems*
  - *Linear time-invariant systems* (LTI)
    - *response to a discrete-time input*
    - *Discrete-time convolution*
    - *properties of LTI systems*
    - *FIR and IIR systems*
    - *(linear) difference equations with constant coefficients*
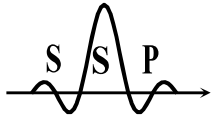
Summary (2/2)

- *Basic signal properties*

  - *Continuous versus discrete*

  - *Periodic versus aperiodic*

  - *Deterministic versus random*

  - *Energy versus power*

  - *Sinusoidal sequences with a prescribed SNR*

- *the auto-correlation and the cross-correlation*

  - *concept and meaning*

  - *definition of the auto-correlation*

  - *definition of the cross-correlation*

  - *auto-correlation and cross-correlation examples*

  - *auto-correlation and cross-correlation properties*

Characterization and representation of discrete signals

- **types of signals and systems**

  - **continuous-time signal** (or analog) = continuous function of independent continuous-time variables

  - **continuous-time systems**: those whose inputs and outputs are continuous

  - **discrete-time signal** = continuous function of independent discrete-time variables

  - **digital signal**: discrete function of independent discrete-time variables

 - **digital systems**: those whose inputs and outputs are discrete

    Despite the fact that digital signal processing presumes digital (*i.e.,* quantized) signals, we will focus mainly on discrete signals and systems and will address quantization separately, when that is required.

Characterization and representation of discrete signals

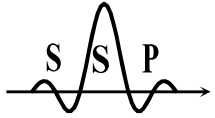- representation of a discrete-time signal

$$x[n], \quad n \in \;]-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty \, [$$

x[n-2]

n integer
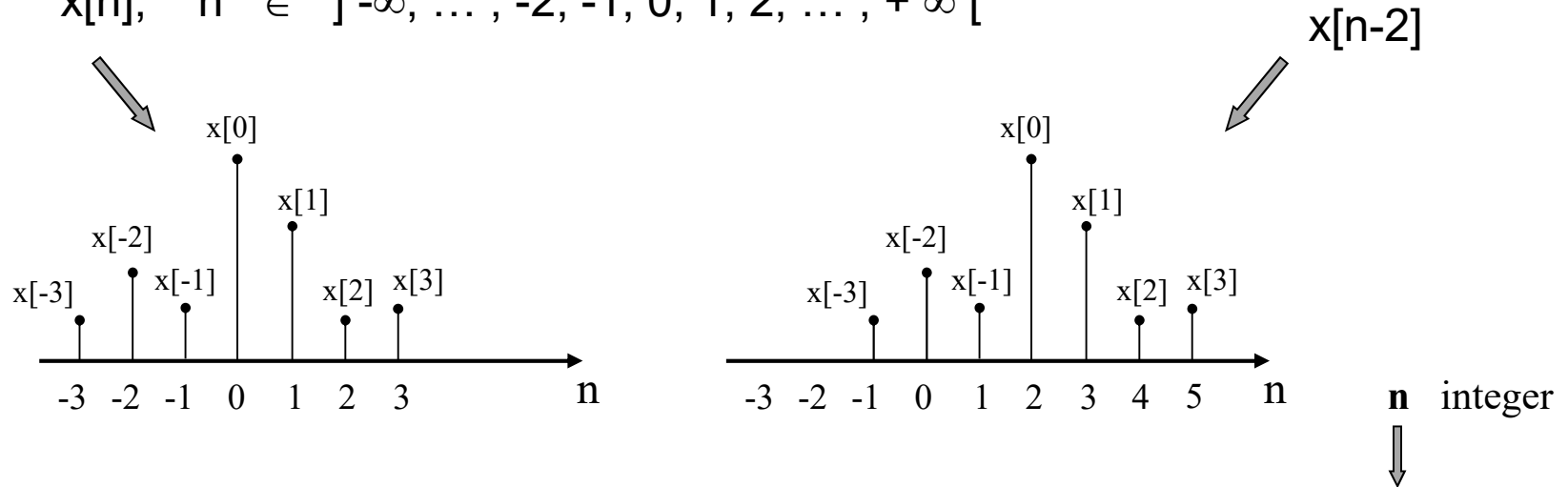
x[n] represents symbolically the ENTIRE discrete signal for $-\infty < n < +\infty$

taking a specific value of n, for example n=2, then x[2] represents the magnitude of the sample of x[n] at position 2 in the sequence of numbers
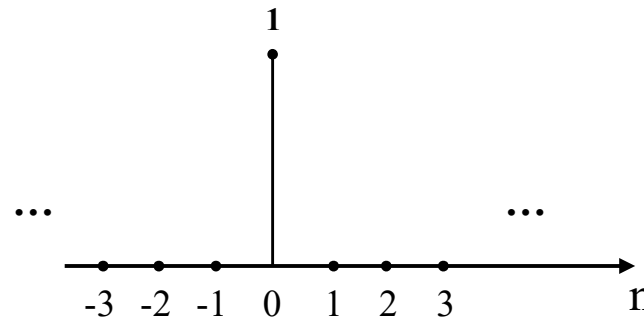
x[n-n$_0$] represents x[n] delayed by n_0 samples
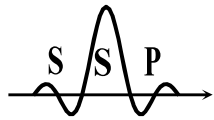
- basic discrete-time signals

  unit impulse (and not "Dirac" impulse! – what is the difference ?)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

- this signal is very important for various reasons, including the possibility to express any discrete-time sequence as a sum of scaled, delayed impulses :
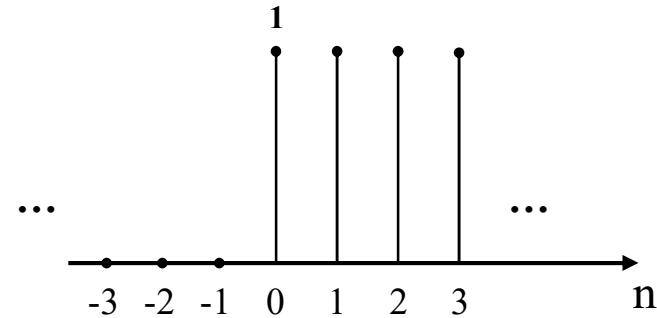
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

<u>unit step</u>

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$
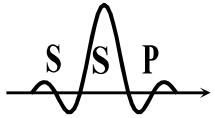
1

... ...

-3 -2 -1   0   1   2   3   n

sum of
delayed impulses

– it is possible to write:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] = \sum_{k=0}^{+\infty} \delta[n-k]$$

accumulated sum
of impulses
till position n

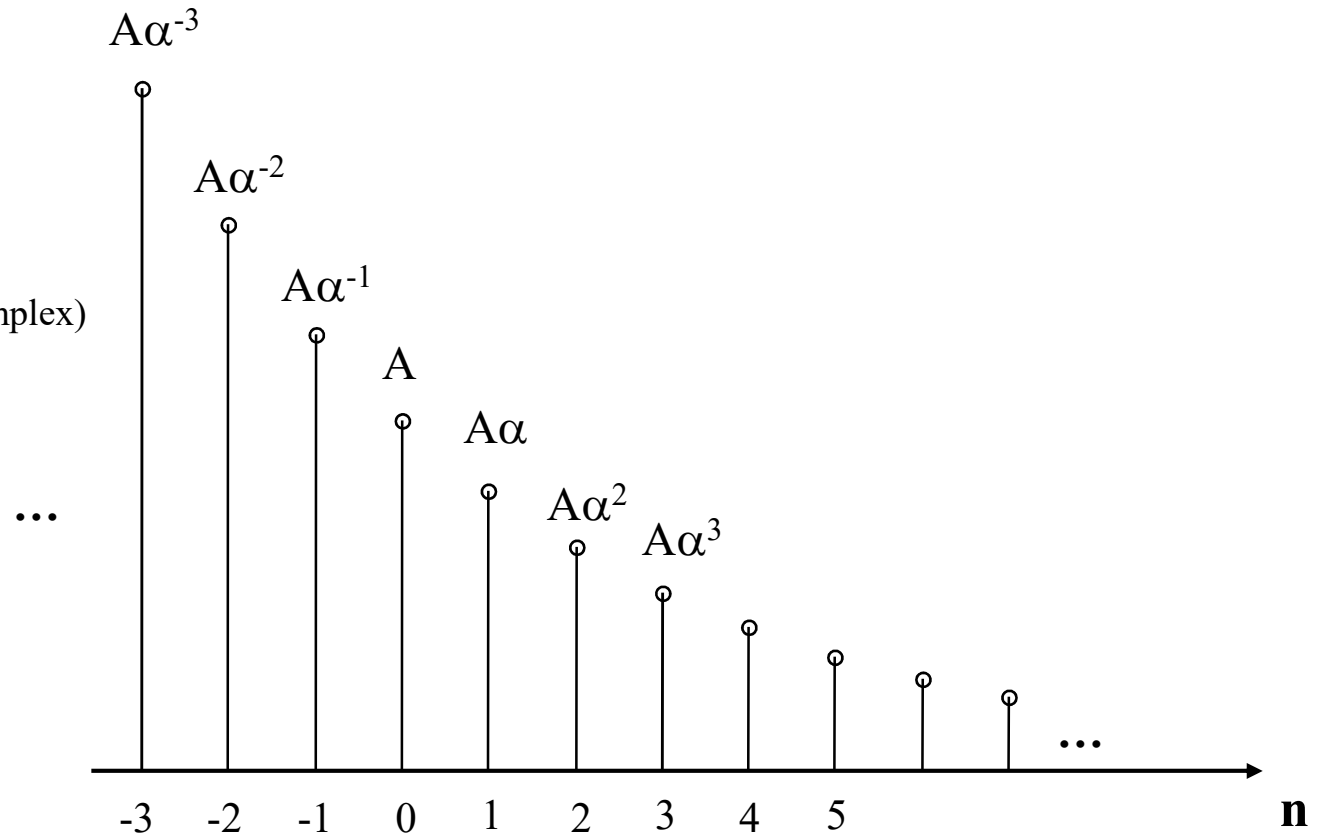NOTE: it is also possible to write $\delta[n] = u[n] - u[n-1]$

exponential sequences

$$x[n] = A\alpha^n$$

(A and/or α may be complex)

$A\alpha^{-3}$

$A\alpha^{-2}$

$A\alpha^{-1}$

A

$A\alpha$

$A\alpha^2$

$A\alpha^3$

...

...

-3 -2 -1 0 1 2 3 4 5

n

important particular case: A=1 and $\alpha=e^{j\omega}$ (unit complex exponential)
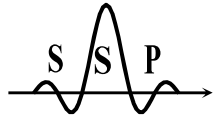
sinusoidal sequences

$$x[n] = A\cos(n\omega_0 + \phi)$$
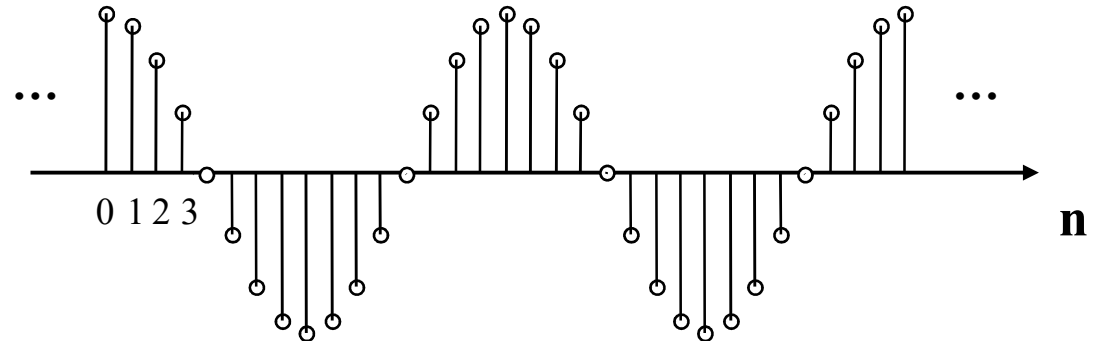
ω_0 - frequency of the sinusoidal sequence [radians]
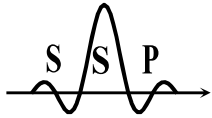
ϕ - phase of the sinusoidal sequence [radians]

NOTE: since $A.\cos[n(\omega_0 + k2\pi) + \phi] = A.\cos[n\omega_0 + \phi]$ with k integer, frequency ω_0 is only defined, for example, in the range $]-\pi, +\pi]$ or $[0, 2\pi[$

combination of sequences

It is common to combine basic discrete-time sequences to express a large variety of signals, e.g. :

$$x[n] = A\alpha^n u[n]$$

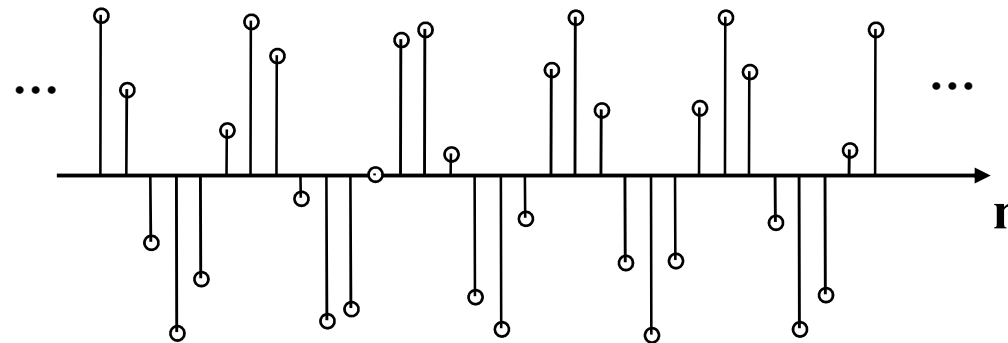$$x[n] = A(u[n] - u[n - n_0])\alpha^n$$

periodic sequences
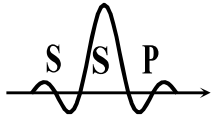
those verifying x[n]=x[n+N], for a given integer N and for any value of n: $-\infty < n < +\infty$

NOTE: since the n index may only take integer values, the counter-intuitive case may occur of a function, such as $\cos(n\omega_0+\phi)$, not showing periodicity in n, for a given ω_0, as for example cos(n) :

QUESTION: Under what condition is that ω_0 insures periodicity in n ?
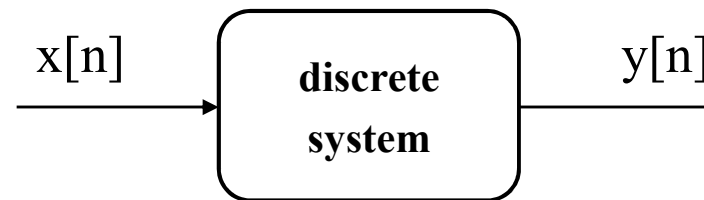
A : $\cos(n\omega_0+\phi) = \cos(n\omega_0+N\omega_0+\phi)$, $\Rightarrow N\omega_0 = k2\pi \Leftrightarrow \omega_0=2\pi k/N$

Characterization and representation of discrete systems

- ## Discrete-time systems

  – they transform an input discrete-time sequence into an output discrete-time sequence

$x[n]$ → [**discrete system**] → $y[n]$

  – Example 1 (delay) : $y[n]=x[n-n_d]$, n_d positive integer = system delay

  – Example 2 (moving average) :
$$y[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} x[n-k]$$

  the output at position n is the average of the (N_1+N_2+1) input samples between position $n-N_2$ and position $n+N_1$
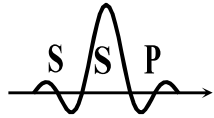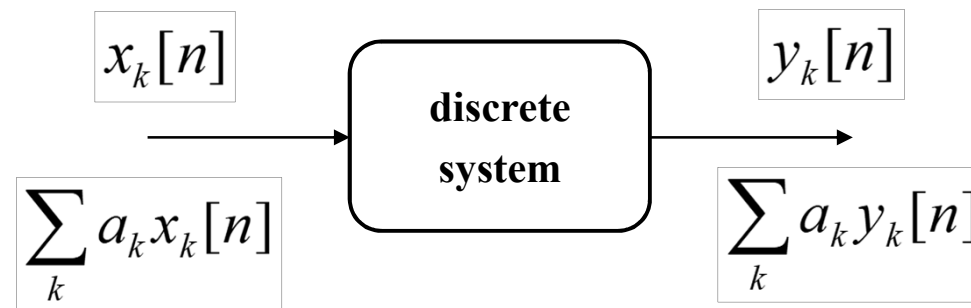
- **Properties of discrete-time systems**

  memory - a memoryless system depends only on the input sample at position n to generate an output sample at the same position

  - example: $y[n] = (x[n])^2$

  linearity – a linear system complies with the superposition principle

  $x_k[n]$ → **discrete system** → $y_k[n]$

  $\sum_k a_k x_k[n]$ → **discrete system** → $\sum_k a_k y_k[n]$

  - example: the "accumulator" function $y[n] = \sum_{k=-\infty}^{n} x[k]$ is linear

  - example: function $y[n] = \log_{10}|x[n]|$ is non-linear
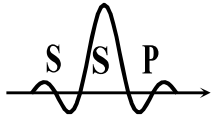
time invariance – a system is time-invariant if a delay of the input sequence gives rise to the same delay of the output sequence
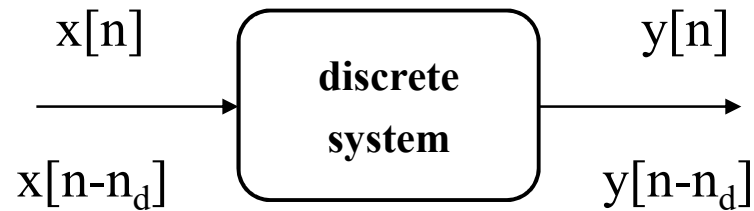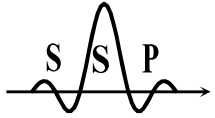
$$x[n] \rightarrow \boxed{\textbf{discrete system}} \rightarrow y[n]$$
$$x[n-n_d] \qquad\qquad\qquad y[n-n_d]$$

- example: the "accumulator" $y[n] = \sum_{k=-\infty}^{n} x[k]$ is time-invariant because if the input samples are delayed by n_0, so are the output samples: $y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$

- example: the decimator system $y[n]=x[nM]$ is not time-invariant

causality – a system is causal (*i.e., it is non-anticipative*) if for any n_0, the output of the system at position $n=n_0$ depends uniquely on the input samples for $n \leq n_0$

- example: $y[n]=x[n]-x[n-1]$ is causal
- example: $y[n]=x[n+1]-x[n]$ is not causal

stability – a system is stable if any bounded input sequence gives rise to an output sequence that is also bounded

$$|x[n]| \leq B_x < \infty, \quad \forall\, n$$

| discrete system |

$$|y[n]| \leq B_y < \infty, \quad \forall\, n$$
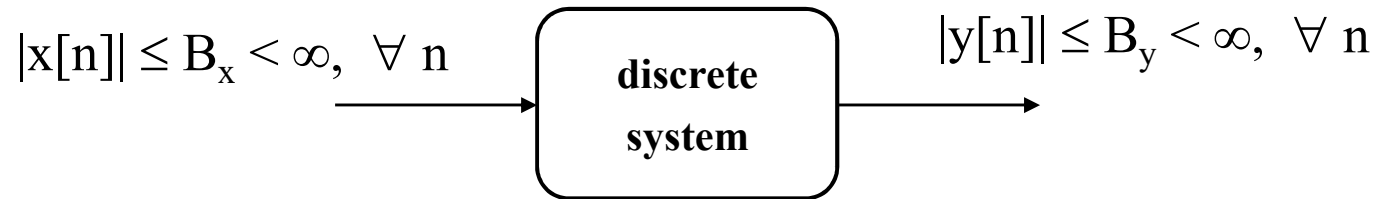
- example: the system $y[n] = (x[n])^2$ is stable

- example: the system $y[n] = \log_{10}|x[n]|$ is not stable (*e.g.*, for x[n]=0)

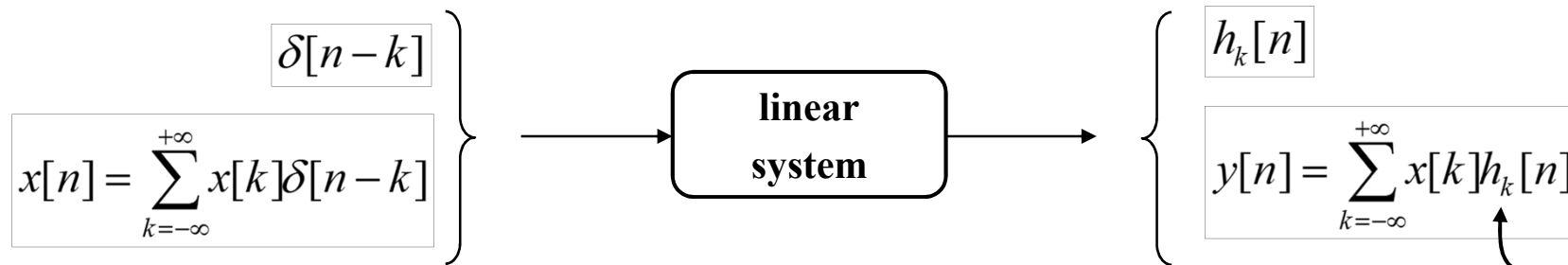- example: the system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not stable (*e.g.*, for x[n]=u[n])

NOTE: a sufficient proof of instability is to find/show a case not complying with the stability condition
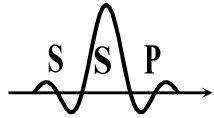
LTI systems: response to a discrete input

- **response to a discrete-time input**

  – if the impulse response is not time-invariant, we have:

$$\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

linear system

$$h_k[n]$$

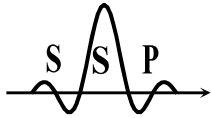$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

this result is of limited usefulness since the response of the system to a linear combination of input impulses is the same combination of the individual responses to the input impulses which may depend on the position of the impulses (time variance)

  – now, if we consider time-invariance:

$$\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

LTI system

$$h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$
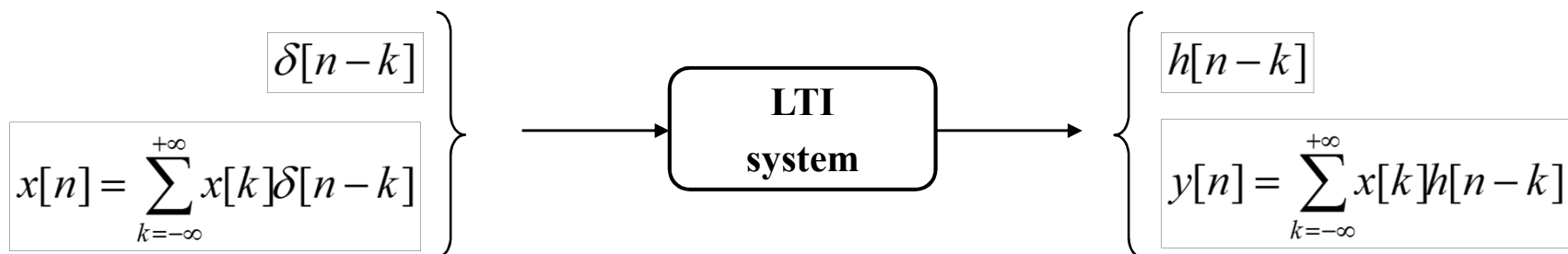
the output of an LTI system is expressed as a function of a single impulse response !

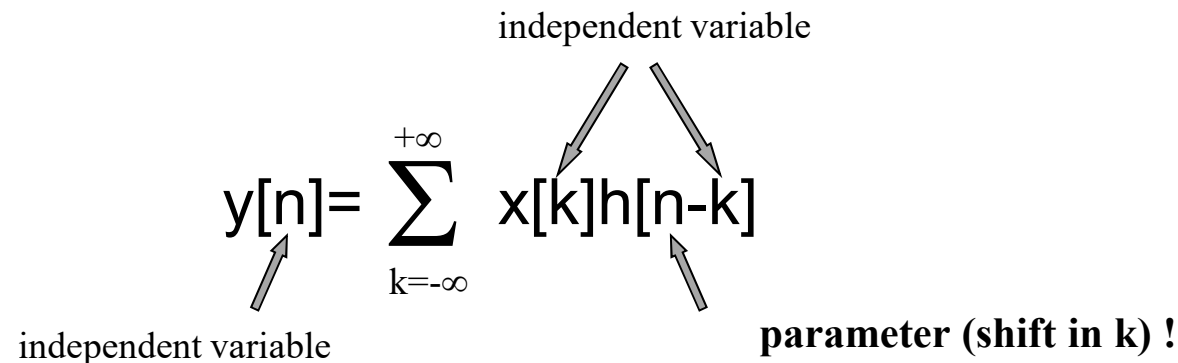© AJF

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Linear time-invariant systems (LTI)

– we may thus say that an LTI system is completely characterized by its impulse response $\Leftrightarrow$ given *h[n]*, it is possible to know the response of the LTI system to *any* input:
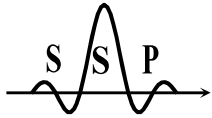
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

this equation consists in the <u>discrete convolution</u> or, in other words, the convolution sum which is reminiscent of the familiar convolution integral for continuous signals:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$
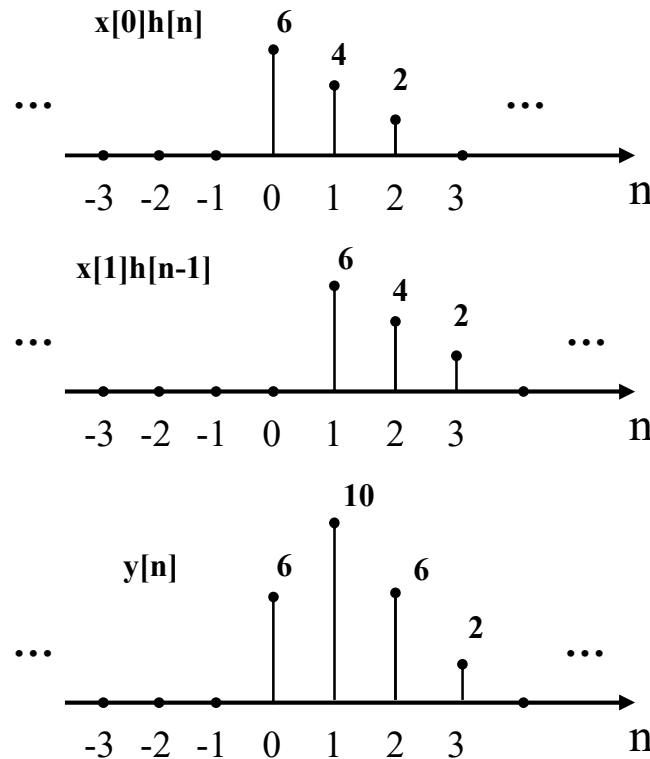
- discrete convolution

independent variable

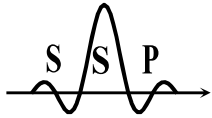$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

independent variable

parameter (shift in k) !

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LTI systems: the discrete convolution

example of the discrete convolution between two sequences:

x[n]

2   2

...   ...

-3 -2 -1   0   1   2   3   n

h[n]   3
                2
                  1
...   ...

-3 -2 -1   0   1   2   3   n

– Method 1: solving by realizing k, we obtain a weighted sum of impulse responses: *y[n]=x[0]h[n]+x[1]h[n-1]*

x[0]h[n]   6
                4
                   2
...   ...

-3 -2 -1   0   1   2   3   n

x[1]h[n-1]   6
                        4
                           2
...   ...

-3 -2 -1   0   1   2   3   n

                    10

y[n]   6   6

                          2
...   ...

-3 -2 -1   0   1   2   3   n

$y[n]=6\delta[n]+10\delta[n-1]+6\delta[n-2]+ 2\delta[n-3]$

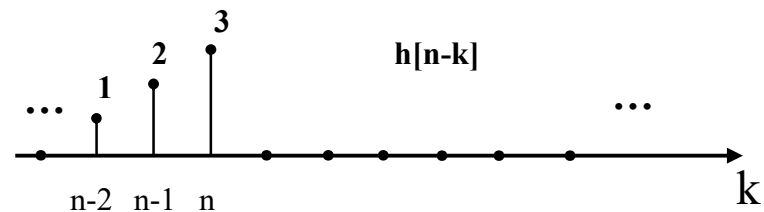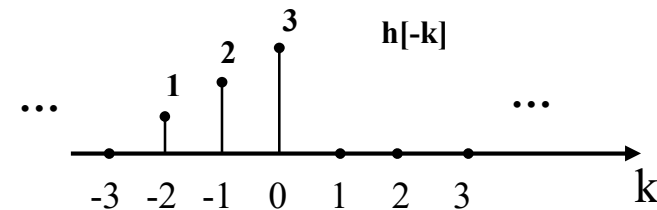© AJF

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LTI systems: the discrete convolution

– Method 2: solving by realizing *n*, we follow a computational procedure similar to the convolution between two continuous signals (one of discrete sequences is time-reversed and is shifted from -∞ till +∞, and for each value of the shift, the accumulation of the sample-to-sample product between this sequence and the other –frozen- signal is computed). Using our example:
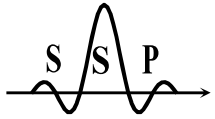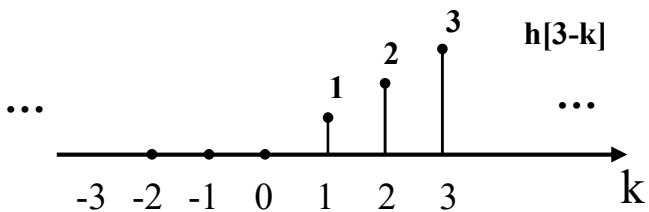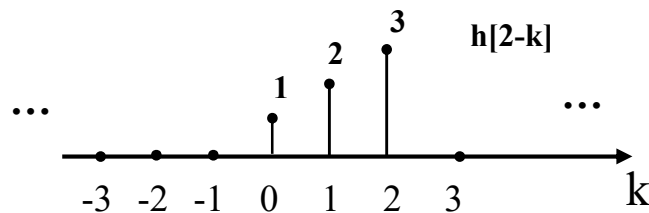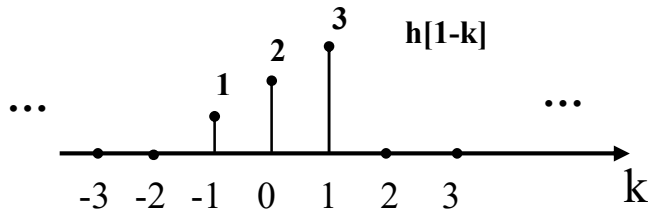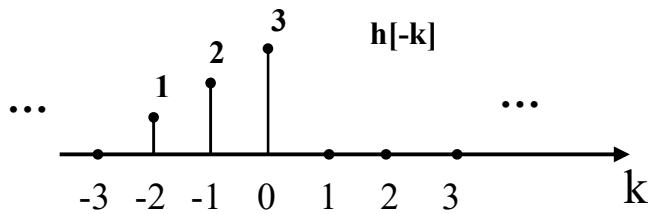

we have:
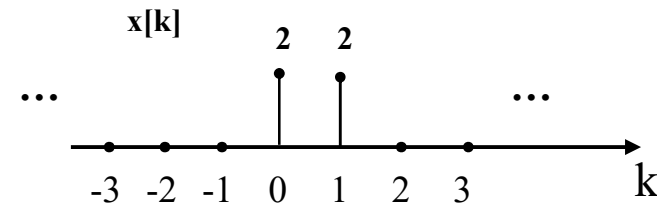
it is clear that
n<0, y[n]=0. For
example, for n=-3 we have:
y[-3]=x[0]h[-3]+x[1]h[-2]=0
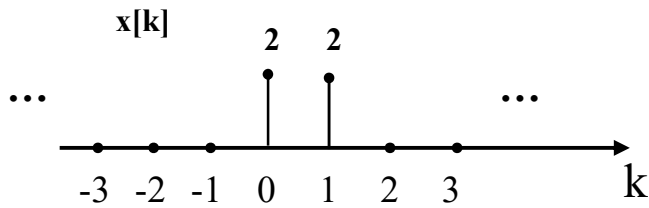
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LTI systems: the discrete convolution

x[k]

-3 -2 -1 0 1 2 3   k

h[-k]

$y[0]=\Sigma\ x[k]h[-k]=x[0]h[0]=6$

-3 -2 -1 0 1 2 3   k

h[1-k]

$y[1]=\Sigma\ x[k]h[1-k]=x[0]h[1]+x[1]h[0]=10$

-3 -2 -1 0 1 2 3   k

h[2-k]

$y[2]=\Sigma\ x[k]h[2-k]=x[0]h[2]+x[1]h[1]=6$

-3 -2 -1 0 1 2 3   k

h[3-k]

$y[3]=\Sigma\ x[k]h[3-k]=x[1]h[2]=2$

-3 -2 -1 0 1 2 3   k

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LTI systems: the discrete convolution

it should be noted that $y[n]=0$ for $n>3$ since $x[k]$ and $h[n-k]$ are not simultaneously different from zero for any value of k.

In summary: $y[n]=6\delta[n]+10\delta[n-1]+6\delta[n-2]+2\delta[n-3]$

Another example: $h[n]=u[n]-u[n-N]$ and $x[n]=\alpha^n u[n]$, with $|\alpha|<1$

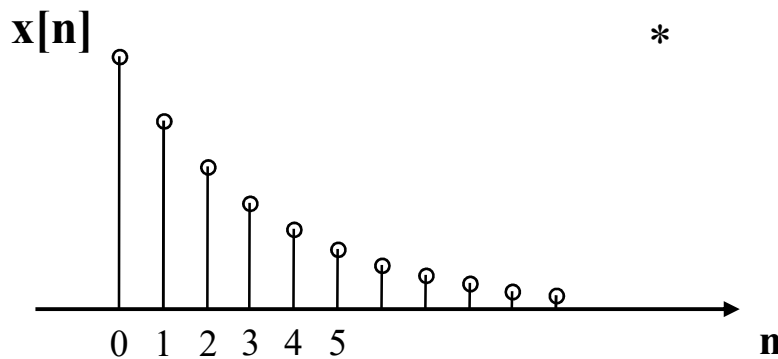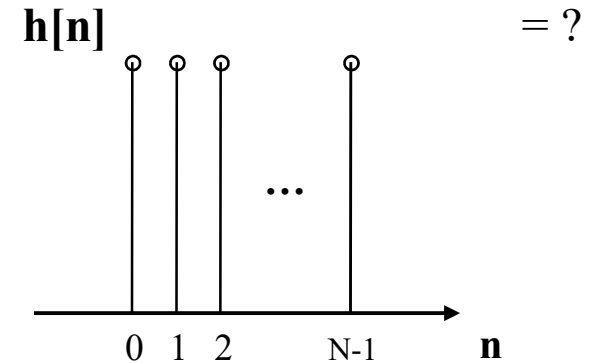QUESTION: which of the previous two methods is more convenient ?

LTI systems: the discrete convolution

$$y[n]= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

x[k]

0 1 2 3 4 5   **k**

h[-k]

...

1-N   0 1 2   **k**

h[N-1-k]

...

0   N-1   **k**
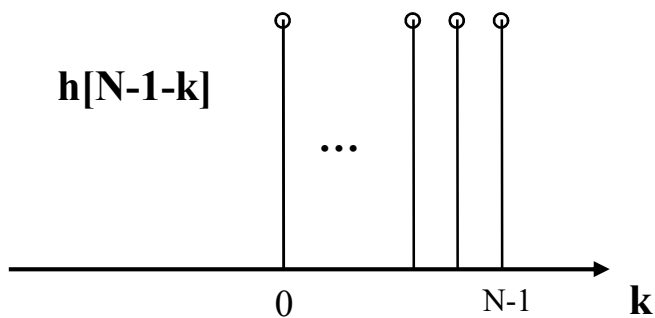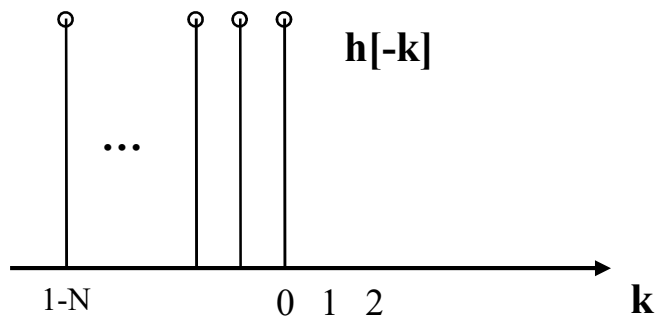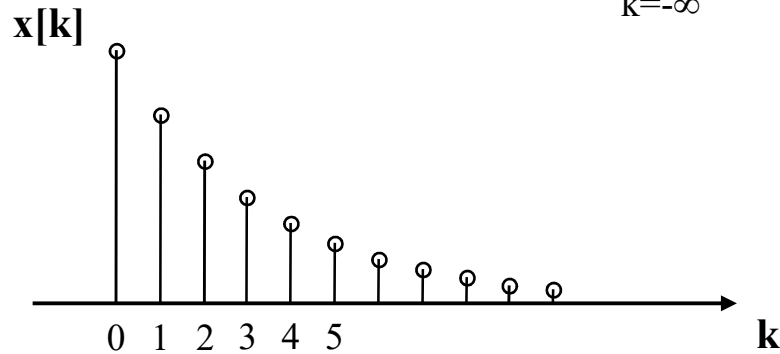
Graphically, it is apparent that we have three intervals for which the y[n] result may be given by a single expression that is valid for all the values of n inside each interval:

Interval 1: n<0 → y[n]=0

Interval 2: 0≤n ≤ N-1

Interval 3: N-1≤ n

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LTI systems: the discrete convolution

- Interval 1: n < 0, y[n]=0

- Interval 2: $0 \leq n \leq N-1$

$u[n-k] \equiv 1$ for $n-k \geq 0 \Leftrightarrow k \leq n$

$u[n-k-N] \equiv 1$ for $n-k-N \geq 0 \Leftrightarrow k \leq n-N$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k u[k]\big(u[n-k]-u[n-k-N]\big) = \sum_{k=0}^{n} \alpha^k$$
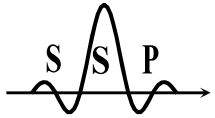
final: $k \geq 0$ && $n-N+1 \leq k \leq n \equiv 0 \leq k \leq n$

(it is easier to conclude graphically)

- Interval 3: $n \geq N-1$

$u[k] \equiv 1$ for $k \geq 0$

$\equiv 1$ for $n-N+1 \leq k \leq n$
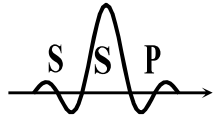
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=n-N+1}^{n} \alpha^k u[k]\big(u[n-k]-u[n-k-N]\big) = \sum_{k=n-N+1}^{n} \alpha^k$$

final: $k \geq 0$ && $n-N+1 \leq k \leq n \equiv n-N-1 \leq k \leq n$

(it is easier to conclude graphically)

- Since the sum of M terms of a geometric series is given by the expression: $G_M = \sum_{k=0}^{M} \alpha^k = \dfrac{1-\alpha^{M+1}}{1-\alpha}$ (for an arithmetic series it would be:)

$$A_M = \sum_{k=1}^{M} k = \frac{M(M+1)}{2}$$

the final result is:

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le N-1 \\ \dfrac{\alpha^{n-N+1}-\alpha^{n+1}}{1-\alpha}, & n \ge N-1 \end{cases}$$
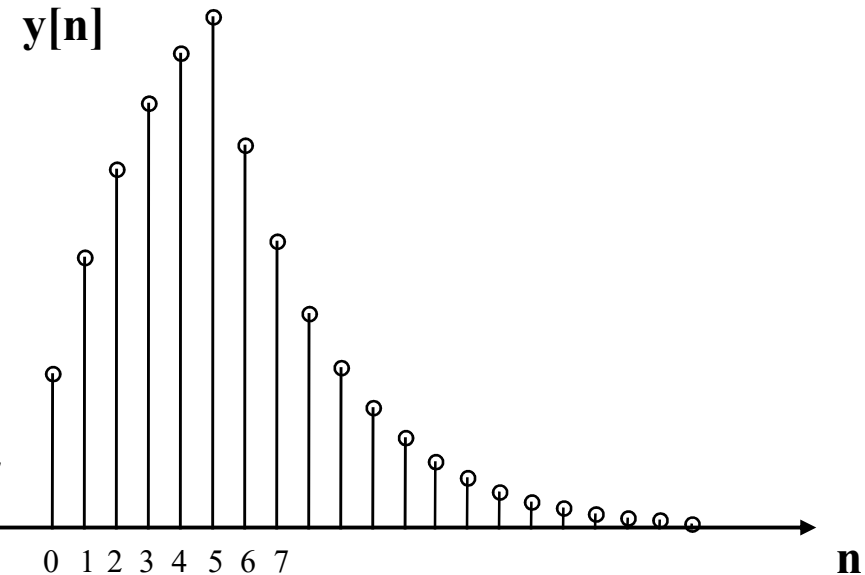
In this illustrative case, what is the value of N ?
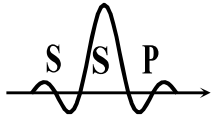
Properties of LTI systems

- Since an LTI system is completely characterized by its impulse response, its properties follow those of the discrete convolution

  - <u>commutative property</u>:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = h[n] * x[n]$$
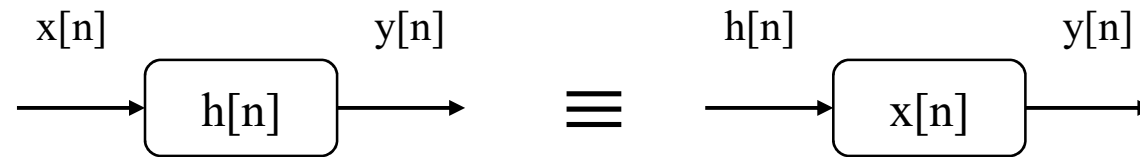
  - <u>distributive property</u> (of the convolution relative to the sum):

$$y[n] = x[n] * \left(h_1[n] + h_2[n] \right) = x[n] * h_1[n] + x[n] * h_2[n]$$
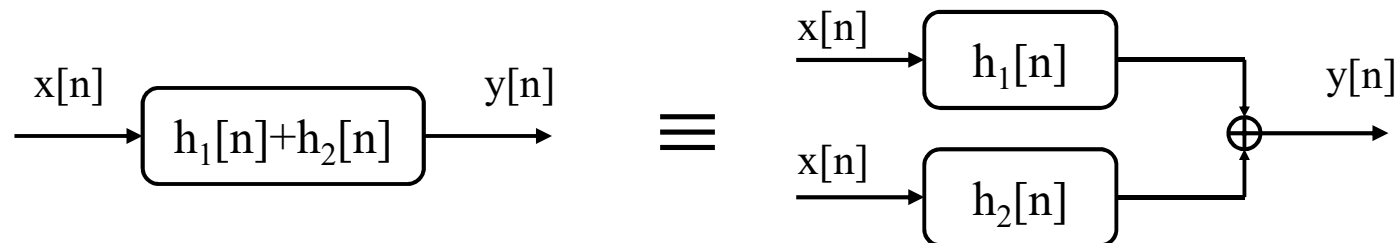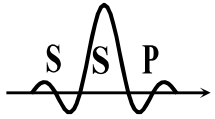
Properties of LTI systems

– <u>series of systems:</u>
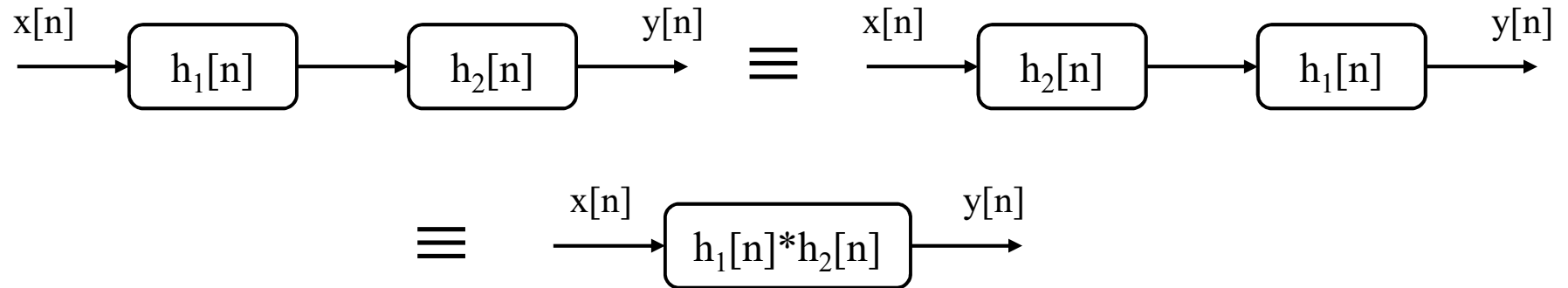
– <u>parallel of systems:</u>
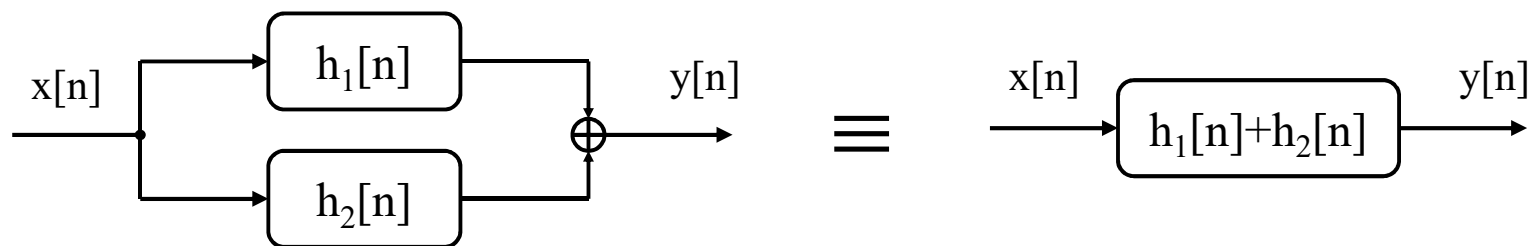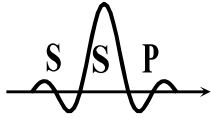
Properties of LTI systems

- condition for the stability of an LTI system: if and only if its impulse response is absolutely summable:

$$S = \sum_{k=-\infty}^{+\infty} |h[k]| \quad < \infty \qquad \text{(necessary and sufficient condition)}$$

- condition for the causality of an LTI system:

$$h[n] = 0, \quad n < 0$$

example: what is the impulse response of each one of the following LTI systems ?

$$y_1[n] = x[n - n_d]$$

$$y_2[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} x[n - k]$$

$$y_3[n] = \sum_{k=-\infty}^{n} x[k]$$

since h[n] = y[n]   we have: $h_1[n] = \delta[n - n_d]$

x[n]=δ[n]

$$h_2[n] = \frac{1}{N_1 + N_2 + 1} \sum_{k=-N_1}^{N_2} \delta[n - k] = \begin{cases} \dfrac{1}{N_1 + N_2 + 1}, & -N_1 \le n \le N_2 \\ 0, & \text{other} \end{cases}$$

$$h_3[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n] = \begin{cases} 1, & n \ge 0 \\ 0 & n < 0 \end{cases}$$
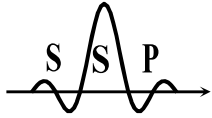
LTI systems: FIR and IIR systems

- According to the number of non-zero samples of its impulse response, an LTI system may be classified as:

  - <u>FIR</u> (finite-duration impulse response): if h[n] has a finite number of non-zero samples
    - NOTE 1: FIR systems are always stable
    - NOTE 2: a non-causal FIR system may be converted into a causal FIR system by adding a suitable delay

  - <u>IIR</u> (infinite-duration impulse response): if h[n] has an infinite number of non-zero samples
    - NOTE: IIR systems may be stable, for example:

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1 \quad \rightarrow \quad S = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-\alpha}$$

    - NOTE: IIR systems may also be unstable, for example:

$$h[n] = u[n] \quad \rightarrow \quad S = \sum_{n=0}^{\infty} 1 = \infty$$

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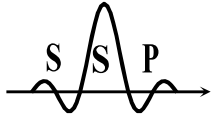
Constant-coefficient difference equations

- Consists in an alternative way (relative to the impulse response) to characterize (although not completely) an LTI (the characterization is only complete if it is added, for example, that the system is causal and starts from rest) relating a combination of delayed inputs with a combination of delayed outputs, and that describes (a realization of) the system:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \quad \Leftrightarrow \quad y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m]$$

  - NOTE 1: this form emphasizes the recursive nature of the relation: the output is obtained after the input sequence is known and after the previous values of the output sequence are known.

  - NOTE 2: if N=0, we have:

$$y[n] = \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m] \quad \Leftrightarrow \quad h[n] = \sum_{m=0}^{M} \frac{b_m}{a_0} \delta[n-m]$$

    which reveals we are dealing with an FIR system, while the more general equation describes an IIR system.
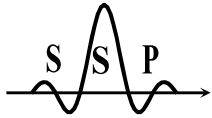
Constant-coefficient difference equations

Example: what is the impulse response of the causal system described by the
difference equation: **y[n]=ay[n-1]+x[n]**

$x[n]$ $y[n]$

delay   **a**
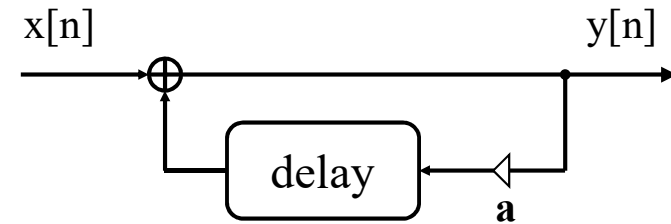
A: considering $x[n]=k\delta[n]$ e admitting that the system starts from rest:

n	x[n]	y[n-1]	y[n]
-1	0	0	0
0	k	0	k
1	0	k	ak
2	0	ak	a^2k
3	0	a^2k	a^3k
4	0	a^3k	a^4k
:	:	:	:
n	0	$a^{n-1}k$	a^nk

this means: $h[n]=a^nu[n]$

NOTE: the same system may be described by different difference equations; a specific
difference equation is indicative of a specific realization of a discrete system among several
possible alternatives (topic to be detailed later on).
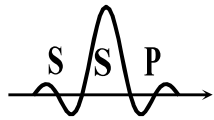
- ## Continuous *versus* discrete

  – a continuous-time signal is a real or complex function of one or more independent variables that, most often, are real-valued, e.g. $x_c(t)$

  - the round brackets reinforce that the independent variable is continuous

  – a discrete-time signal is a real or complex function of one or more independent variables that can take on integer values only, e.g. $x[n]$

  - the square brackets denote that the enclosed variable is discrete
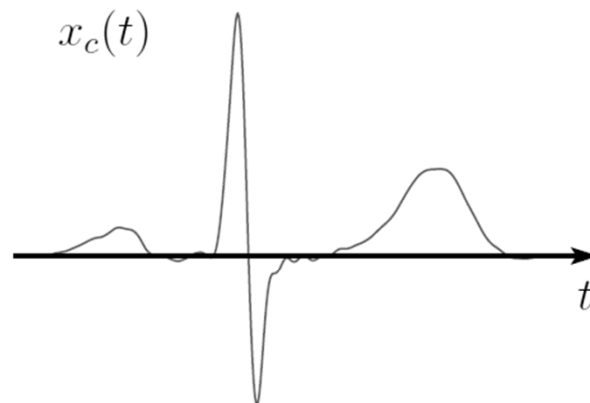
Fig. 1.16 A continuous-time signal representing the bioelectric voltage due to one cycle of the heart beat.
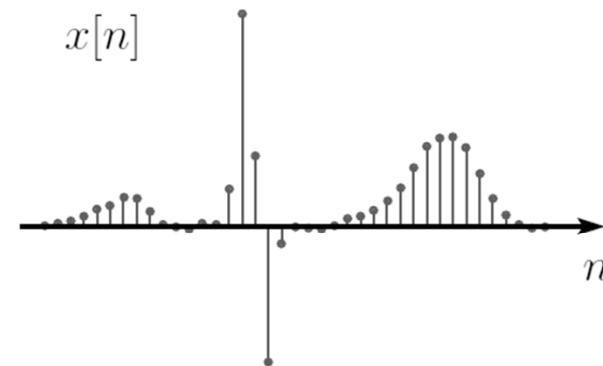
Fig. 1.17 A discrete-time version of the continuous-time signal in Fig. 1.16

- Periodic *versus* aperiodic

  – a periodic discrete-time signal is one whose structure or pattern repeats in n for some finite period N, i.e.

$$x[n] = x[n + N], \qquad N \in \mathbb{Z}\backslash\{0\}, \qquad \forall\, n \in \mathbb{Z}$$
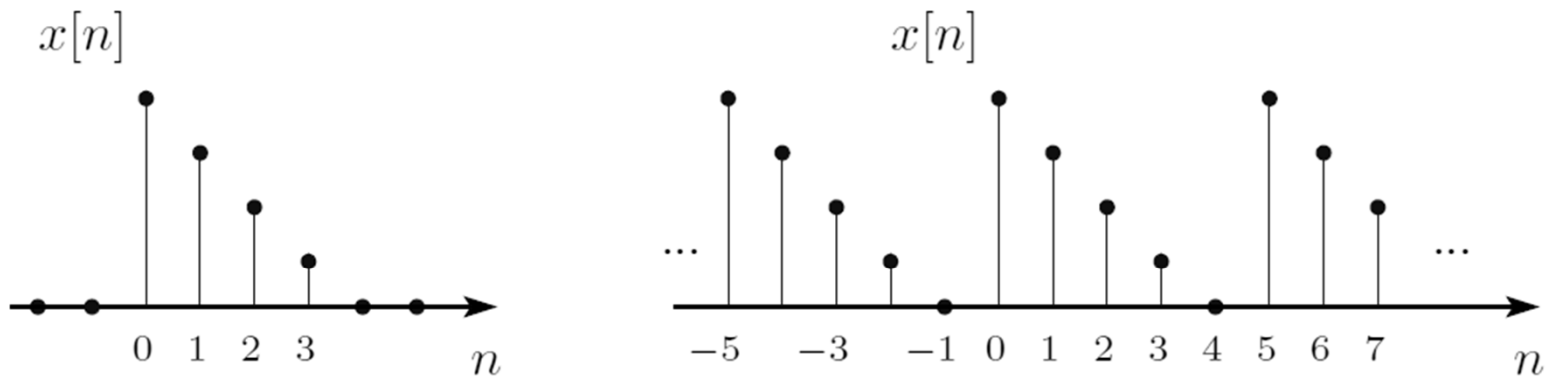
Fig. 1.20 Illustration of an aperiodic discrete sequence (figure on the left) and a periodic discrete sequence whose period is $N = 5$ (figure on the right).

Question: if a periodic signal is obtained by repeating periodically an aperiodic one, what condition makes that the aperiodic signal is recognizable in the periodic signal ?

- **Deterministic *versus* random**
    - a deterministic signal is specified by a mathematical function, an algorithmic procedure, or computational method, which completely determines the signal value at any point in the discrete-time domain n
    - quite often, these deterministic signal representation alternatives are replaced by a lookup table (LUT) when trading computation for memory is desired
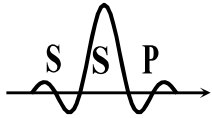   - a notable feature of a deterministic signal is that it is predictable
    - if a signal is deterministic, so are its statistical properties, for example, the probability density function (PDF) of the signal amplitudes

    - a random signal is characterized by unpredictability and uncertainty concerning the value of each sample in a discrete sequence
    - instead of being governed by a deterministic rule, the realization of each sample in a random sequence is governed by a probabilistic model underlying the stochastic process that generates that sequence
    - although a random signal is unpredictable at sample level, certain practical assumptions, such as stationarity, make that 'latent' probabilistic attributes underlying a random signal, are predictable
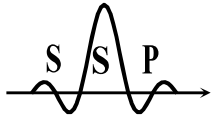
- ## Deterministic *versus* random
  - a stationary random signal may exhibit a specific PDF, for example, uniform, of Gaussian
  - these possibilities are illustrated next and have been created using the Matlab functions `rand()` and `randn()`
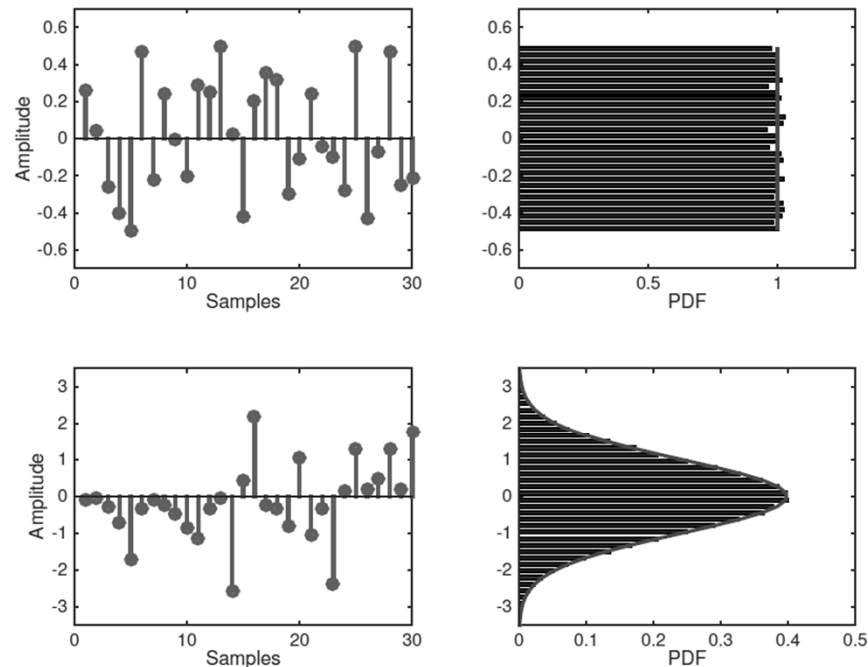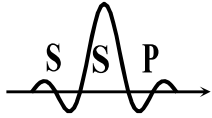

Fig. 1.23 Representation of the first 30 samples of two random sequences, one having a uniform PDF between -0.5 and 0.5 (top left) and another having a Gaussian PDF with unit variance (bottom left). The corresponding PDF are inferred on the right-hand side by means of histograms. The exact PDF models are also represented as a solid (magenta) lines.

Basic signal properties

- **Energy *versus* power**
  - a discrete-time signal $x[n]$ is classified as an energy signal if its energy (E) is finite, i.e. if

$$E = \sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

  - any finite-length sequence is always classified as an energy signal as long as its samples have a finite magnitude
  - a discrete-time signal $x[n]$ is classified as a power signal if its energy is infinite but a finite result is obtained when the energy of an arbitrary large number of samples is divided by the number of samples

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 < \infty$$

  - in the case of an *N*-periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 < \infty$$

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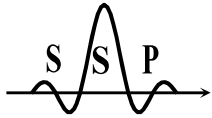
- • Sinusoidal sequences with a prescribed SNR
  - – frequently, discrete-time test signals need to be generated that have a specific level of noise contamination; most often, these test signals consist of a single complex exponential, or a single sinusoidal sequence, and the noise consists of complex or real-valued Gaussian noise

 - – the severity of the noise contamination is objectively determined by the ratio between the average power of the signal, which we represent as P_S, and the average power of the noise, which we represent as P_N

  - – that ratio is typically evaluated on a logarithmic scale in tenths of a unit called Bel, which is usually abbreviated to deciBel, or dB; thus, this scale expresses a proportion which is called the Signal-to-Noise Ratio (SNR) and is defined as

$$SNR = 10 \log_{10} \frac{P_S}{P_N}$$
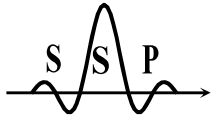
- **Sinusoidal sequences with a prescribed SNR**
  - the following example illustrates the case of a real sinusoid that is contaminated by Gaussian noise according to a prescribed SNR; the result is confirmed numerically in Matlab

Example 1.12. Create in Matlab a real-valued sinusoidal sequence having frequency $\omega_0 = 0.0123$ rad. and magnitude $A = 1$. Create a noise vector such that when it is added to the sinusoidal sequence the resulting SNR is 30 dB. Validate numerically the Matlab code using 10^4 samples.

Examples 1.8 and 1.9 have shown that the average power of a real-valued sinusoid (or co-sinusoid) having magnitude A is given by $A^2/2$. Example 1.11 has also shown that the average power of (zero-mean) Gaussian noise is given by σ^2. Generating a noise vector complying with a desired SNR just involves using the randn$(\cdot)$ Matlab function with the appropriate σ parameter. Thus, we find first σ based on (1.55):

$$\sigma = \frac{A}{10^{SNR/20}\sqrt{2}}.$$

After both discrete sequences are created, a numerical validation is easily achieved by estimating the corresponding average power according to Eq. (1.16.4) or Eq. (1.52). A possible Matlab implementation is listed next.
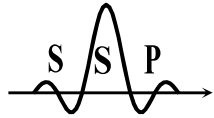
```
N=1E4; n=[0:N-1].';
omega=0.0123; A=1; signal=A*sin(omega*n);
SNR=30; sigma=A/(sqrt(2)*10^(SNR/20));
noise=sigma*randn(N,1);
Ps=mean((abs(signal)).^2);
Pn=mean((abs(noise)).^2);
10*log10(Ps/Pn)

ans = 30.06
```
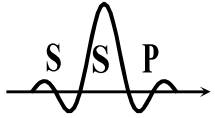
- **concept and meaning**

  – the auto-correlation and the cross-correlation are two important discrete-time signal processing functions that consist in specific forms of the discrete-time convolution

  – they are designed to evaluate the similarity between two discrete-time signals, or waveforms

  – if the two waveforms are based on the same discrete-time signal, then the function is called auto-correlation, whereas if the two waveforms are based on different discrete-time signals, then the function is called cross-correlation
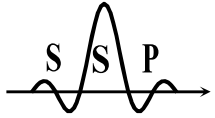
- **definition of the auto-correlation**
  - the auto-correlation ($r_x[\ell]$) assesses the similarity between between a finite-energy reference signal, $x[k]$, and the conjugate of its shifted version $x^*[k - \ell]$, where ℓ represents a discrete-time shift, also commonly referred to as lag

  - it is obtained as the discrete convolution between $x[\ell]$ and the conjugate of its time-reversed version, $x^*[-\ell]$:

$$r_x[\ell] \triangleq x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]x^*[-(\ell - k)] = \sum_{k=-\infty}^{+\infty} x[k]x^*[k - \ell]$$

  - The auto-correlation may be regarded as a self-similarity measure for a given discrete-time shift, if the self-similarity is strong, then the auto-correlation exhibits a high absolute value; conversely, if the self-similarity is weak, then the auto-correlation exhibits a small absolute value, nearing zero in the case of strong dissimilarity
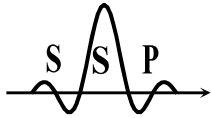
- **definition of the cross-correlation**

  – the cross-correlation ($r_{xy}[\ell]$) assesses the similarity between between a finite-energy reference signal, $x[k]$, and $y^*[k - \ell]$ that represents the conjugate of another discrete-time signal, y[k], that is also affected by a lag ℓ

  – it is obtained as the discrete convolution between $x[\ell]$ and $y^*[-\ell]$:

  $$r_{xy}[\ell] \triangleq x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]y^*[-(\ell - k)] = \sum_{k=-\infty}^{+\infty} x[k]y^*[k - \ell]$$

  – if the similarity between the two waveforms is strong for a given lag, then the cross-correlation function exhibits a high absolute value for that lag, however, if the similarity is weak, then the cross-correlation exhibits a small absolute value; when it nears zero, one may say the signals are approximately uncorrelated

- auto-correlation and cross-correlation examples
  - a special waveform $x[n]$ is designed such that its auto-correlation consists of a single impulse, i.e. $r_x[\ell] = \delta[\ell]$
  - a second waveform y$[n]$ is generated that consists of a noisy version of $x[n+3]$, i.e. $y[n] = x[n+3] + v[n]$, where $v[n]$ is a random sequence that is not correlated to $x[n]$
  - both $r_x[\ell]$ and $r_{xy}[\ell]$ are represented in the following figure
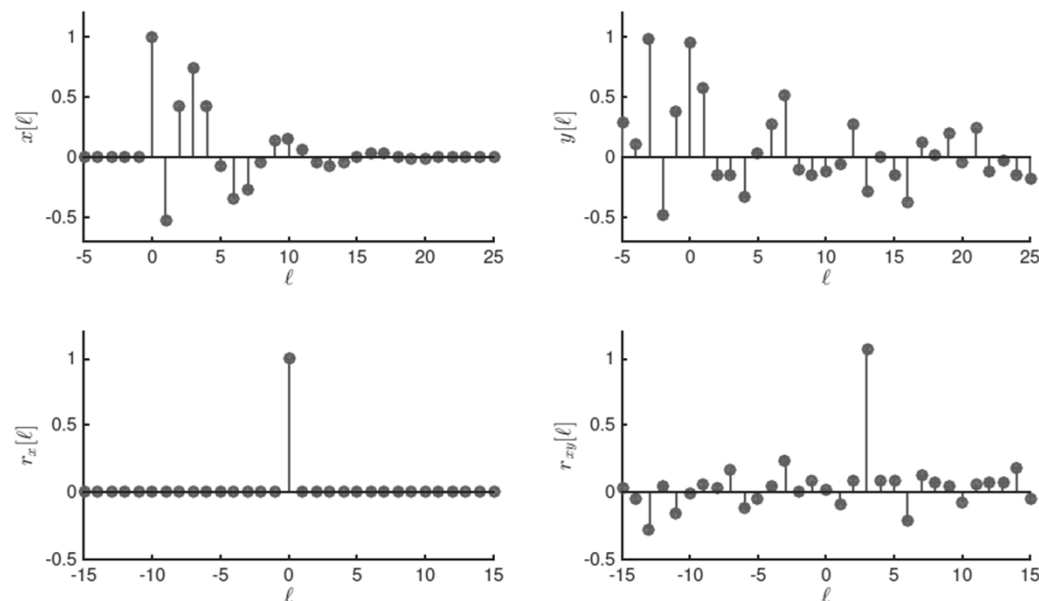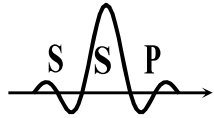
Fig. 1.76 Example of a waveform $x[\ell]$ (top left figure), its auto-correlation function $r_x[\ell]$ (bottom left figure), a waveform $y[\ell]$ consisting of noisy version of $x[\ell+3]$ (top right figure), and the cross-correlation function $r_{xy}[\ell]$ (bottom right figure).
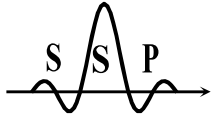
- **auto-correlation and cross-correlation properties**
  - in the case of the auto-correlation, it can be easily shown that if the lag is zero, then we obtain the energy of the signal

$$r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = E_x$$

  - the auto-correlation is conjugate-symmetric, i.e. $r_x[\ell] = r_x^*[-\ell]$

  - the cross-correlation verifies $r_{xy}[\ell] = r_{yx}^*[-\ell]$
    - which does **not** mean conjugate-symmetry

  - it can also be shown that $\left| r_{xy}[\ell] \right| \leq \sqrt{E_x E_y} = \sqrt{r_x[0] r_y[0]}$

  - and, as a particular case, the auto-correlation is upper bounded by the signal energy:

$$|r_x[\ell]| \leq r_x[0] = E_x$$

- **auto-correlation and cross-correlation properties**
  - the results in the previous slide can be used to normalize both auto-correlation and cross-correlation functions, which leads to $\rho_x[\ell]$ and $\rho_{xy}[\ell]$:

$$\rho_x[\ell] = \frac{r_x[\ell]}{r_x[0]} , \qquad\qquad -1 < \rho_x[\ell] \leq 1, \ \forall \ \ell \in \mathbb{Z}$$

$$\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_x[0]r_y[0]}} , \qquad -1 \leq \rho_{xy}[\ell] \leq 1, \ \forall \ \ell \in \mathbb{Z}$$

Question: the previous results have been developed for energy signals, how should they be adapted for power signals ?