Overview

- *discrete-time random processes*
  - filtering random processes
  - spectral factorization
  - Wold decomposition theorem
  - special types of processes
    - autoregressive moving-average processes (ARMA)
    - autoregressive processes (AR)
    - moving-average processes (MA)
    - harmonic processes

- *filtering random processes*

  – linear shift-invariant filters are frequently used in applications involving signal band limiting, signal detection and estimation, deconvolution, signal representation and synthesis

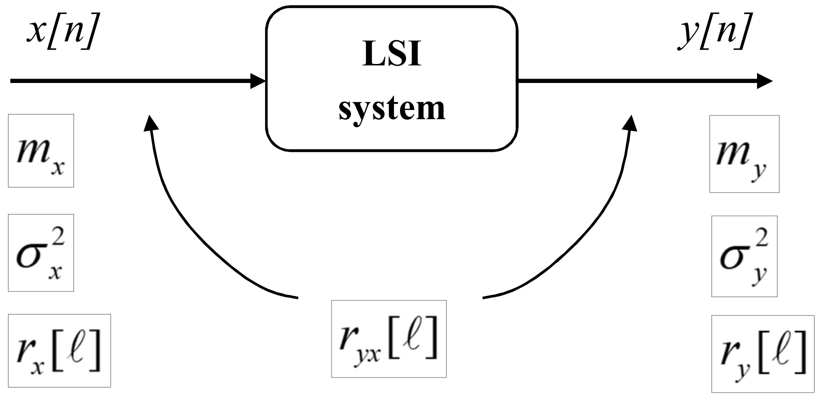
  – as the input may be a random process, it is important to determine how the statistics change from input to output

if $x[n]$ is a WSS random process with mean m_x and autocorrelation $r_x[\ell]$ and if $h[n]$ is the impulse response of a stable LTI filter

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

review of discrete-time random processes

- *filtering random processes*



if the impulse response of the LSI system is *h[n]*

- how is the output mean expressed as a function of the input mean and *h[n]* ?
- how is the cross-correlation expressed as a function of the input autocorrelation and *h[n]* ?
- how is the output autocorrelation expressed as a function of the input correlation and *h[n]* ?
- what is the deterministic autocorrelation of the system ?

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- *filtering random processes*

the mean of the output *y[n]* is

$$E\{y[n]\}=\sum_{k=-\infty}^{\infty}h[k]E\{x[n-k]\}=\sum_{k=-\infty}^{\infty}h[k]m_x=m_x\sum_{k=-\infty}^{\infty}h[k]e^{j0k}=m_xH\left(e^{j0}\right)=m_xH(1)$$

i.e. the mean of the output is the mean of the input scaled by the frequency response of the filter at DC (i.e. at ω=0)

on the other hand, the correlation between the output and input is given by

$$r_{yx}[n,n-\ell]=E\{y[n]x^*[n-\ell]\}=E\left\{\sum_{k=-\infty}^{\infty}h[k]x[n-k]x^*[n-\ell]\right\}=$$

$$=\sum_{k=-\infty}^{\infty}h[k]E\{x[n-k]x^*[n-\ell]\}=\sum_{k=-\infty}^{\infty}h[k]r_x[n-k-n+\ell]=$$

$$=\sum_{k=-\infty}^{\infty}h[k]r_x[\ell-k]=h[\ell]*r_x[\ell]=r_{yx}[\ell]$$

Note: $r_{xy}[\ell]=h^*[-\ell]*r_x[\ell]=r_{yx}^*[-\ell]$
but $r_{xy}[\ell]\neq r_{xy}^*[-\ell]$

i.e. it results from the convolution between the autocorrelation of the input and the impulse response of the filter and depends only on the lag ℓ

- *filtering random processes*

using the previous result, we may find the autocorrelation of the output:

$$r_y[n,n-\ell]=E\{y[n]y^*[n-\ell]\}=E\left\{y[n]\sum_{k=-\infty}^{\infty}h^*[k]x^*[n-\ell-k]\right\}$$

$$r_y[n,n-\ell]=E\left\{\sum_{k=-\infty}^{\infty}h^*[k]y[n]x^*[n-\ell-k]\right\}$$

by setting $m=\ell+k$ and changing the summation index to m (since $\ell<\infty$)

$$r_y[n,n-\ell]=E\left\{\sum_{m=-\infty}^{\infty}h^*[m-\ell]y[n]x^*[n-m]\right\}=\sum_{m=-\infty}^{\infty}h^*[m-\ell]E\{y[n]x^*[n-m]\}=$$

$$=\sum_{m=-\infty}^{\infty}h^*[m-\ell]E\{y[n]x^*[n-m]\}=\sum_{m=-\infty}^{\infty}h^*[m-\ell]r_{yx}[m]=\sum_{m=-\infty}^{\infty}r_{yx}[m]h^*[-(\ell-m)]=$$

$$=\sum_{m=-\infty}^{\infty}r_{yx}[m]h^*[-(\ell-m)]=r_{yx}[\ell]*h^*[-\ell]$$

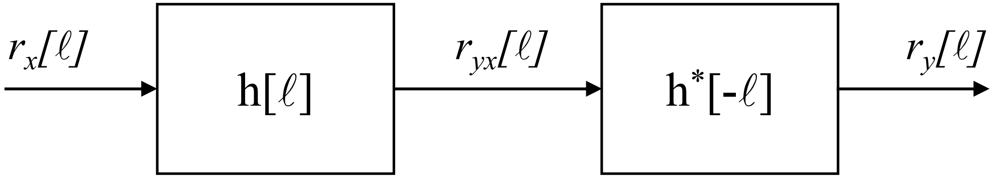
now, since $r_{yx}[\ell]=h[\ell]*r_x[\ell]$

we finally find $r_y[\ell]=r_x[\ell]*h[\ell]*h^*[-\ell]=r_x[\ell]*r_h[\ell]$

review of discrete-time random processes

- *filtering random processes*

this result defines a quantity $r_h[\ell]$, which is the deterministic autocorrelation of the LSI system, and states that the autocorrelation of the output is simply the result of the convolution between the autocorrelation of the input and the deterministic autocorrelation of the system, in general:



– regarding the output variance, from the previous slide we have:

$$r_y[\ell] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k] r_x[m-k] h^*[m-\ell]$$

if the input is zero mean, then $\sigma_y^2 = r_y[0] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k] r_x[m-k] h^*[m]$

and if it is white noise: $\sigma_y^2 = r_y[0] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k] \sigma_x^2 \delta[m-k] h^*[m] = \sigma_x^2 \sum_{k=-\infty}^{\infty} \left|h[k]\right|^2$

review of discrete-time random processes

in particular, if *h[n]* is finite in length (and zero outside the interval [0, N-1]), using the result of slide 43 (week 2), the output variance σ_y^2 may also be written as ($\mathbf{R}_x$ is the autocorrelation matrix and **h** is vector of filter coefficients)

$$\sigma_y^2 = \sum_{m=0}^{N-1}\sum_{k=0}^{N-1} h^*[m]\, r_x[m-k]\, h[k] = \mathbf{h}^H \mathbf{R}_x \mathbf{h}$$

also, by noting that

$$h[n] \xleftrightarrow{F} H\left(e^{j\omega}\right)$$
$$h^*[n] \xleftrightarrow{F} H^*\left(e^{-j\omega}\right)$$
$$h^*[-n] \xleftrightarrow{F} H^*\left(e^{j\omega}\right)$$

then

$$r_{yx}[\ell] = r_x[\ell] * h[\ell] \xleftrightarrow{F} P_{yx}\left(e^{j\omega}\right) = P_x\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$
$$r_y[\ell] = r_x[\ell] * h[\ell] * h^*[-\ell] \xleftrightarrow{F} P_y\left(e^{j\omega}\right) = P_x\left(e^{j\omega}\right) \left|H\left(e^{j\omega}\right)\right|^2$$

in terms of the Z-transform:

$$h[n] \xleftrightarrow{Z} H(z)$$
$$h^*[n] \xleftrightarrow{Z} H^*\left(z^*\right)$$
$$h^*[-n] \xleftrightarrow{Z} H^*\left(1/z^*\right)$$

Note: for a stable system the unit circle $z=e^{j\omega}$ lies within the ROCs of H(z) and $H(z^{-1})$

and therefore:

$$r_{yx}[\ell] = r_x[\ell] * h[\ell] \xleftrightarrow{Z} P_{yx}(z) = P_x(z) H(z)$$
$$r_y[\ell] = r_x[\ell] * h[\ell] * h^*[-\ell] \xleftrightarrow{Z} P_y(z) = P_x(z) H(z) H^*\left(1/z^*\right)$$

Example: finding the autocorrelation at the output of a filter with transfer function $H(z)$ when its input is white noise with variance σ_x^2

$$H(z)=\frac{1}{1-0.25z^{-1}},\quad \text{causal}$$

$$r_x[\ell]=\sigma_x^2\delta[n] \quad \overset{Z}{\longleftrightarrow} \quad P_x(z)=\sigma_x^2$$

$$P_y(z)=P_x(z)H(z)H^*(1/z^*)=\sigma_x^2\frac{1}{1-0.25z^{-1}}\frac{1}{1-0.25z}$$

partial fraction expansion of $P_y(z)$ leads to

$$P_y(z)=\sigma_x^2\frac{z^{-1}}{(1-0.25z^{-1})(z^{-1}-0.25)}=\sigma_x^2\left(\frac{16/15}{1-0.25z^{-1}}+\frac{4/15}{z^{-1}-0.25}\right)=$$

$$=\sigma_x^2\left(\frac{16/15}{1-0.25z^{-1}}+\frac{16/15}{1-4z^{-1}}\right)$$

the inverse Z-transform yields

$$r_y[\ell]=\sigma_x^2\frac{16}{15}\left(\frac{1}{4}\right)^{\ell}u[\ell]+\sigma_x^2\frac{16}{15}(4)^{\ell}u[-\ell-1]=\sigma_x^2\frac{16}{15}\left(\frac{1}{4}\right)^{|\ell|}$$

- *spectral factorization*
  - The power spectrum $P_x(e^{j\omega})$ of a wide-sense stationary process is a real-valued, positive, and periodic function of ω. If $P_x(e^{j\omega})$ is a continuous function of ω (i.e. it does not contain impulses), then it may be seen as describing the output of a stable discrete-time filter when its input is white noise having a variance of σ_x^2, i.e. $P_x(e^{j\omega})$ may be factored as

$$P_x(z) = \sigma_0^2 H(z) H^*(1/z^*)$$

In fact, if $P_x(z) = \sum_{\ell=-\infty}^{+\infty} r_x[\ell] z^{-\ell}$ is the power spectrum of a WSS stationary process $x[n]$, and if ln $P_x(e^{j\omega})$ is analytic in an annulus $\rho<|z|<1/\rho$, i.e. ln $P_x(e^{j\omega})$ and all of its derivatives are continuous functions of z and can be expanded in a Laurent series:

$$\ln P_x(z) = \sum_{k=-\infty}^{+\infty} c[k] z^{-k}$$

Note: in the cepstral domain, the multiplicative factors H(z) and $H^*(1/z^*)$ are additively separable due to the natural logarithm of $P_x(e^{j\omega})$

where $c[k]$ are the coefficients of the expansion (these coefficients may also be regarded as the cepstrum of the sequence $r_x[\ell]$), then

$$P_x(z) = \exp\left\{ \sum_{k=-\infty}^{+\infty} c[k] z^{-k} \right\}$$

review of discrete-time random processes

It should be noted that the coefficients *c[k]* are conjugate symmetric, in fact it is possible to write

$$\ln P_x\left(e^{j\omega}\right)=\sum_{k=-\infty}^{+\infty} c[k]e^{-j\omega k} \quad \overset{F}{\longleftrightarrow} \quad c[k]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln P_x\left(e^{j\omega}\right)e^{j\omega k}d\omega$$

and because $P_x(e^{j\omega})$ is real and so ln $P_x(e^{j\omega})$, it follows that the *c[k]* coefficients are conjugate symmetric, i.e. $c[k]=c^*[-k]$ and thus

$$P_x(z)=\exp\{c[0]\}\exp\left\{\sum_{k=1}^{+\infty}c[k]z^{-k}\right\}\exp\left\{\sum_{k=-1}^{-\infty}c[k]z^{-k}\right\}$$

however, since

$$\exp\left\{\sum_{k=-1}^{-\infty}c[k]z^{-k}\right\}=\exp\left\{\sum_{k=1}^{+\infty}c[-k]z^{k}\right\}=\exp\left\{\sum_{k=1}^{+\infty}c^*[-k]\left(1/z^*\right)^{-k}\right\}^*=\exp\left\{\sum_{k=1}^{+\infty}c[k]\left(1/z^*\right)^{-k}\right\}^*$$

we may write: $P_x(z)=\exp\{c[0]\}Q(z)Q^*(1/z^*)=\sigma_0^2\,Q(z)Q^*(1/z^*)$

which reflects the spectral factorization of $P_x(e^{j\omega})$. A process which can be factored this way is called a <u>*regular process*</u>.

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Since *Q(z)* is the z-transform of a causal and stable sequence *c[k]*, it may be expanded into a power series in the form

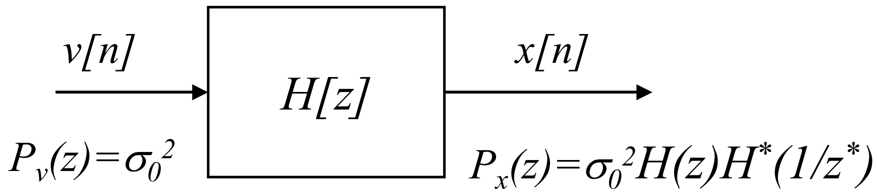
$$Q(z) = \exp\left\{ \sum_{k=1}^{+\infty} c[k]\, z^{-k} \right\} = 1 + q[1]z^{-1} + q[2]z^{-2} + \ldots$$

Note: *Q(z)* is a monic polynomial, i.e. one for which the coefficient of the zeroth-order term is equal to one.

Also, since *Q(z)* and ln *Q(z)* are both analytic for $|z|>\rho$, then *Q(z)* is a *minimum phase filter*. This means that for a rational function of z, all poles and zeros lie inside the unit circle of the z-plane. As a consequence, the inverse filter *1/Q(z)* is also a causal and stable filter (and is a *minimum phase filter*).

Properties of a regular process

- *innovations representation*: any regular process may be realized as the output of a causal and stable filter that is driven by white noise having a variance of σ_0^2,



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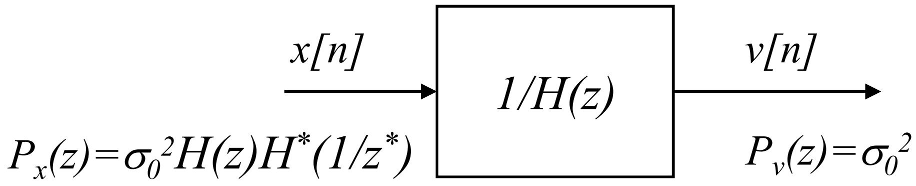
Properties of a regular process (*cont.*)

- *innovations process*: if $x[n]$ is filtered with $1/H(z)$, then the output is white noise with variance σ_0^2, i.e. the inverse filter $1/H(z)$ is a *whitening filter*



If $P_x(z)$ is rational function $P_x(z) = \dfrac{N(z)}{D(z)}$ then it may be factored as

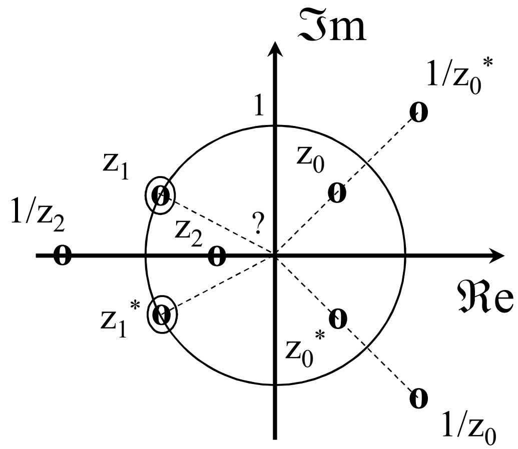
$$P_x(z) = \sigma_0^2\, Q(z) Q^*(1/z^*) = \sigma_0^2\, \frac{B(z)}{A(z)} \frac{B^*(1/z^*)}{A^*(1/z^*)}$$

where $B(z)$ and $A(z)$ are monic polynomials having all the roots within the unit circle. It should be noted that since the autocorrelation function of a WSS random process is conjugate symmetric, then the power spectrum $P_x(e^{j\omega})$ is a real function of ω and $P_x(z)=P_x^*(1/z^*)$

review of discrete-time random processes

– *Why for any rational power spectrum, factorization is possible*: It should be noted that since the autocorrelation function of a WSS random process is conjugate symmetric, then the power spectrum $P_x(e^{j\omega})$ is a real function of ω and $P_x(z)=P_x^*(1/z^*)$, this means that for each zero (or pole) in $P_x(z)$, there will be a matching zero (or pole) at the conjugate reciprocal location. Another (equivalent) way of looking at this is: since $P_x(e^{j\omega})$ is a real function of ω, then the combined filter $Q(z)Q^*(1/z^*)$, whose region of convergence is $\rho<|z|<1/\rho$ (i.e. a ring) and includes the unit circumference (i.e. it is stable), has poles and zeros which occur in reciprocal and complex conjugate pairs,

- Illustration of the symmetry conditions for the zeros of the power spectrum



review of discrete-time random processes

- *Wold decomposition theorem*
  - Any WSS random process may be decomposed into the sum of two orthogonal processes, a regular process $x_r[n]$ and a predictable (i.e. deterministic) process $x_p[n]$

$$x[n] = x_p[n] + x_r[n]$$
$$E\{x_p[m]x_r^*[n]\} = 0$$

  - corollary: general form for the power spectrum of a WSS process is

$$P_x(e^{j\omega}) = P_{xr}(e^{j\omega}) + \sum_{k=1}^{N} \alpha_k \delta(\omega - \omega_k)$$

    - $P_{xr}(e^{j\omega})$ is the *continuous part* of the spectrum corresponding to the regular process
    - the predictable part of the process gives rise to a line spectrum

review of discrete-time random processes

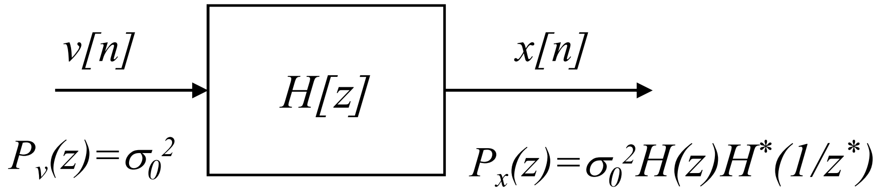
- *special types of random processes*
  - those processes that may be generated by filtering white noise with a linear shift-invariant filter that has a rational system function, these include the autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) processes

Note: parametric pole-zero models describe a system with a finite number of parameters.

- *autoregressive moving average processes (ARMA)*
  - white noise $v[n]$ is filtered with a causal linear shift-invariant filter having a rational system function with p poles and q zeros

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b[k] z^{-k}}{1 + \sum_{k=1}^{p} a[k] z^{-k}}$$

  - since



$$P_v(z) = \sigma_0^2 \qquad P_x(z) = \sigma_0^2 H(z) H^*(1/z^*)$$

review of discrete-time random processes

- *autoregressive moving average processes (ARMA)*
  - assuming that *H(z)* is stable, the output process *x[n]* is wide-sense stationary and if $P_v(z)=\sigma_v^2$ then

$$P_x(z) = \sigma_v^2 \frac{B(z)B^*(1/z^*)}{A(z)A^*(1/z^*)}$$

in terms of ω

$$P_x\left(e^{j\omega}\right) = \sigma_v^2 \frac{\left|B\left(e^{j\omega}\right)\right|^2}{\left|A\left(e^{j\omega}\right)\right|^2}$$

- the power spectrum of an ARMA process contains *2p* poles and *2q* zeros with the reciprocal symmetry relationships discussed in slide 13

since *x[n]* and *v[n]* are related by

$$x[n] + \sum_{k=1}^{p} a[k]x[n-k] = \sum_{k=0}^{q} b[k]v[n-k]$$

review of discrete-time random processes

then, in order to find the relation between the autocorrelation of the output and the cross-correlation between output and input, we multiply both sides of the equation by $x^*[n-\ell]$ and take the expectation:

$$x[n]x^*[n-\ell]+\sum_{k=1}^{p}a[k]x[n-k]x^*[n-\ell]=\sum_{k=0}^{q}b[k]v[n-k]x^*[n-\ell]$$

$$E\{x[n]x^*[n-\ell]\}+\sum_{k=1}^{p}a[k]E\{x[n-k]x^*[n-\ell]\}=\sum_{k=0}^{q}b[k]E\{v[n-k]x^*[n-\ell]\}$$

$$r_x[\ell]+\sum_{k=1}^{p}a[k]r_x[\ell-k]=\sum_{k=0}^{q}b[k]r_{vx}[\ell-k]$$

in order to find $r_{vx}[\ell-k]$, since $x[n]=\sum_{m=-\infty}^{\infty}h[m]v[n-m]$

Note: *h[n]* may be found since the coefficients of the transfer function of the filter are known.

$$r_{vx}[\ell-k]=E\{v[n-k]x^*[n-\ell]\}=E\left\{v[n-k]\sum_{m=-\infty}^{\infty}h^*[m]v^*[n-\ell-m]\right\}$$

$$r_{vx}[\ell-k]=\sum_{m=-\infty}^{\infty}h^*[m]E\{v[n-k]v^*[n-\ell-m]\}=\sum_{m=-\infty}^{\infty}h^*[m]r_v[\ell+m-k]$$

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but since the input is white noise, i.e. $r_v[\ell+m-k]=\sigma_v^2\delta[\ell+m-k]$

and thus $r_{vx}[\ell-k]=\sigma_v^2 h^*[k-\ell]$

Note: this result could also be found by using the result of slide 4

$$r_{xv}[\ell]=\sum_{m=-\infty}^{\infty} h[m]r_v[\ell-m]$$

and also the fact that

$$r_{vx}[\ell]=r_{xv}^*[-\ell]$$

thus

$$r_{vx}[\ell-k]=\sum_{m=-\infty}^{\infty} h^*[m]r_v^*[k-\ell-m]$$

and since

$$r_v^*[k-\ell-m]=\sigma_v^2\delta[k-\ell-m]$$

therefore $r_{vx}[\ell-k]=\sigma_v^2 h^*[k-\ell]$

which leads to

$$r_x[\ell]+\sum_{k=1}^{p} a[k]r_x[\ell-k]=\sigma_v^2\sum_{k=0}^{q} b[k]h^*[k-\ell]$$

Assuming that *h[n]=0, n<0*, i.e. that *h[n]* is causal, then

$$r_x[\ell]+\sum_{k=1}^{p} a[k]r_x[\ell-k]=\begin{cases}\sigma_v^2 c[\ell] & ; \quad 0\le\ell\le q\\ 0 & ; \quad \ell>q\end{cases}$$

with

$$c[\ell]=\sum_{k=\ell}^{q} b[k]h^*[k-\ell]=\sum_{k=0}^{q-\ell} b[k+\ell]h^*[k]$$

Yule-Walker equations (important in e.g. signal modeling and spectrum estimation)

- relate filter coefficients and the autocorrelation sequence
- allow to extrapolate the autocorrelation sequence (for $\ell\ge p$) if p≥q and if the values $r_x[0]$, $r_x[1]$,…, $r_x[p-1]$ are known

review of discrete-time random processes

- *autoregressive processes (AR)*
  - result for ARMA(p,q) processes when q=0, i.e. when B(z)=b[0]:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b[0]}{1 + \sum_{k=1}^{p} a[k] z^{-k}}$$

if $r_v[\ell] = \sigma_v^2 \delta[\ell]$ (i.e. $P_v(z) = \sigma_v^2$) then

$$P_x(z) = \sigma_v^2 \frac{|b[0]|^2}{A(z) A^*(1/z^*)}$$

and in terms of ω

$$P_x\left(e^{j\omega}\right) = \sigma_v^2 \frac{|b[0]|^2}{\left|A\left(e^{j\omega}\right)\right|^2}$$

  - the power spectrum of an AR(p) process contains 2p poles and no zeros (except those at z=0 and z=∞)

review of discrete-time random processes

- *autoregressive processes (AR)*
  – since for an AR(p) process *b[0]=h[0]*, then *c[0]* in the Yule-Walker equations (slide 18) is *c[0]= b[0]b*[0]*=|*b[0]*|^2, thus

$$r_x[\ell]+\sum_{k=1}^{p} a[k]\,r_x[\ell-k]=\sigma_v^2\left|b[0]\right|^2\delta[\ell] \quad ; \quad \ell \geq 0$$

which, in matrix form, makes that the Yule-Walker equations become

$$\begin{bmatrix} r_x[0] & r_x[-1] & \cdots & r_x[-p] \\ r_x[1] & r_x[0] & \cdots & r_x[1-p] \\ \vdots & \vdots & \ddots & \vdots \\ r_x[p] & r_x[p-1] & \cdots & r_x[0] \end{bmatrix}\begin{bmatrix} 1 \\ a[1] \\ \vdots \\ a[p] \end{bmatrix} = \sigma_v^2\left|b[0]\right|^2\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

these equations allow e.g. to

- find the coefficients *a[k]* from the autocorrelation sequence $r_x[\ell]$
- to generate the autocorrelation sequence from a given set of filter coefficients

- *autoregressive processes (AR)*
  - Example: finding the coefficients *a[k]* from the autocorrelation sequence $r_x[\ell]$ of a real-valued AR(1) process

assuming $\sigma_v^2 = 1$ and using the property for real-processes that $r_x[\ell] = r_x[-\ell]$, the the Yule-Walker equations reduce to

$$
\begin{aligned}
r_x[0] + a[1]\, r_x[1] &= |b[0]|^2 \\
r_x[1] + a[1]\, r_x[0] &= 0
\end{aligned}
$$

from which we obtain

$$
a[1] = -\frac{r_x[1]}{r_x[0]}
$$

$$
|b[0]|^2 = \frac{r_x^2[0] - r_x^2[1]}{r_x[0]}
$$

and thus

$$
H(z) = \frac{B(z)}{A(z)} = \frac{b[0]}{1 + a[1]z^{-1}} = \frac{\sqrt{r_x[0]\left(r_x^2[0] - r_x^2[1]\right)}}{r_x[0] - r_x[1]z^{-1}}
$$

- *autoregressive processes (AR)*
  - Example: generating the autocorrelation sequence from a given set of filter coefficients (first-order AR process)

since $r_x[\ell]+a[1]r_x[\ell-1]=\sigma_v^2\left|b[0]\right|^2\delta[\ell] \quad ; \quad \ell\geq 0$

assuming $\sigma_v^2=1$ and using the property for real-processes that $r_x[\ell]=r_x[-\ell]$, the the Yule-Walker equations reduce to

$$\begin{bmatrix} 1 & a[1] \\ a[1] & 1 \end{bmatrix}\begin{bmatrix} r_x[0] \\ r_x[1] \end{bmatrix}=\begin{bmatrix} \left|b[0]\right|^2 \\ 0 \end{bmatrix}$$

from which we obtain

$$r_x[0]=\frac{b^2[0]}{1-a^2[1]}$$

$$r_x[1]=-\frac{a[1]b^2[0]}{1-a^2[1]}$$

also, as in general $r_x[\ell]=-a[1]r_x[\ell-1] \quad ; \quad \ell\geq 1$

then $r_x[\ell]=\dfrac{b^2[0]}{1-a^2[1]}\left(-a[1]\right)^{\ell} \quad ; \quad \ell\geq 0$ or ($r_x[\ell]=r_x[-\ell]$) $r_x[\ell]=\dfrac{b^2[0]}{1-a^2[1]}\left(-a[1]\right)^{|\ell|}$

- *moving average processes (MA)*
  - result for ARMA(p,q) processes when p=0, i.e. when A(z)=1:

$$H(z)=\sum_{k=0}^{q} b[k]z^{-k}$$

in this case, *x[n]* is generated by filtering white noise with an FIR filter
if $r_v[\ell]=\sigma_v^2\delta[\ell]$ (i.e. $P_v(z)=\sigma_v^2$) then, an MA(q) process has power spectrum

$$P_x(z)=\sigma_v^2\,B(z)B^*(1/z^*)$$

and in terms of ω

$$P_x\left(e^{j\omega}\right)=\sigma_v^2\left|B\left(e^{j\omega}\right)\right|^2$$

  - the power spectrum of an MA(q) process contains 2q zeroes and no poles (except those at z=0 and z=∞)
  - using the results of slide 18 and noting that *h[n]=b[n]*

$$c[\ell]=\sum_{k=\ell}^{q} b[k]b^*[k-\ell]=\sum_{k=0}^{q-\ell} b[k+\ell]b^*[k]$$

- *moving average processes (MA)*

then

$$r_x[\ell]=\sigma_v^2\sum_{k=0}^{q-\ell} b[k+\ell]\,b^*[k]=\sigma_v^2 b[\ell]*b^*[-\ell] \quad ; \quad 0\le \ell \le q$$

Note: this result is in line with those in slide 5.

considering that $r_x[\ell]=r_x[-\ell]$, we may also write

$$r_x[\ell]=\sigma_v^2\sum_{k=0}^{q-|\ell|} b[k+|\ell|]\,b^*[k] \quad ; \quad 0\le |\ell| \le q$$

which helps to emphasize that if $\ell \notin$ [-q, q] , then $r_x[\ell]=0$ also, $r_x[\ell]$ depends non-linearly on the MA parameters *b[k]*, e.g. for an MA(2) process

$$r_x[0]=\sigma_v^2\left(b^2[0]+b^2[1]+b^2[2]\right)$$

$$r_x[1]=\sigma_v^2\left(b^*[0]b[1]+b^*[1]b[2]\right)$$

$$r_x[2]=\sigma_v^2 b^*[0]b[2]$$

which makes it more difficult (than for an AR process) to estimate the MA parameters.

– MA processes are characterized by slowly changing functions of frequency that have sharp nulls in the spectrum if $P_x(z)$ contains zeros that are close to the unit circle.

review of discrete-time random processes

- *Harmonic processes*
  - provide useful representations for random processes that arise in applications such as array processing, when the signals contain periodic components
  - an example of a WSS harmonic process is the random phase sinusoid

$$x[n] = A\sin\left(n\omega_0 + \phi\right)$$

where *A* and ω_0 are constants and ϕ is a random variable uniformly distributed in the range [-π, π[; as a result, the autocorrelation of *x[n]* is periodic with frequency ω_0

$$r_x[\ell] = \frac{A^2}{2}\cos\left(\ell\omega_0\right)$$

and the power spectrum is

$$P_x\left(e^{j\omega}\right) = \pi\frac{A^2}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

if the amplitude is also a random variable that is uncorrelated with ϕ, then

$$r_x[\ell] = \frac{1}{2}E\left\{A^2\right\}\cos\left(\ell\omega_0\right)$$

- *Harmonic processes*
  - higher-order harmonic processes may be formed from a sum of random phase sinusoids:

$$x[n]=\sum_{k=1}^{L} A_k \sin\left(n\omega_k+\phi_k\right)$$

Note: the harmonic process is predictable because any given realization is a sinusoidal sequence with fixed amplitude, frequency and phase.

if ϕ_k are pairwise independent and if the random variables ϕ_k and A_k are uncorrelated, the autocorrelation sequence is

$$r_x[\ell]=\frac{1}{2}\sum_{k=1}^{L} E\left\{A_k^2\right\}\cos\left(\ell\omega_k\right)$$

and the power spectrum is

$$P_x\left(e^{j\omega}\right)=\frac{\pi}{2}\sum_{k=1}^{L} E\left\{A_k^2\right\}\left[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)\right]$$

  - In the case of the sum of L complex uncorrelated harmonic processes

$$x[n]=\sum_{k=1}^{L} A_k \exp\left[j\left(n\omega_k+\phi_k\right)\right]$$

$$r_x[\ell]=\sum_{k=1}^{L} E\left\{\left|A_k\right|^2\right\}\exp\left(j\ell\omega_k\right)$$

$$P_x\left(e^{j\omega}\right)=2\pi\sum_{k=1}^{L} E\left\{\left|A_k\right|^2\right\}\delta(\omega-\omega_k)$$