Formulae Sheet (part 1)

Discrete convolution	$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$
Fourier Transform	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$ $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$
Z-Transform	$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}, Roc = R_x$ $\quad x[n] = \frac{1}{2\pi j}\oint_C X(z)\, Z^{n-1} dz$ $x[n] = a^n u[n] \leftrightarrow X(z) = \frac{1}{1-az^{-1}}, \quad \lvert z\rvert > \lvert a\rvert$ $x[n] = -a^n u[-n-1] \leftrightarrow X(z) = \frac{1}{1-az^{-1}}, \quad \lvert z\rvert < \lvert a\rvert$ $x[n-n_0] \leftrightarrow z^{-n_0}X(z), \quad Roc \equiv R_X$ $z_0^n x[n] \leftrightarrow X(z/z_0), \quad Roc \equiv \lvert z_0\rvert R_X$ $x^*[n] \leftrightarrow X^*(z^*), \quad Roc \equiv R_X$ $x[-n] \leftrightarrow X(z^{-1}), \quad Roc \equiv 1/R_X$ $x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z), \quad Roc \equiv R_{X1} \cap R_{X2}$
random processes	$m_X = E\{x\}$ $\quad$ mean square value: $E\{\lvert x\rvert^2\}$ $\sigma_X^2 = E\{\lvert x-m_X\rvert^2\} = E\{\lvert x\rvert^2\} - \lvert m_X\rvert^2$ $r_{XY} = E\{xy^*\}$ $c_{XY} = E\{(x-m_X)(y-m_Y)^*\} = E\{xy^*\} - m_X m_Y^*$ $\rho_{XY} = \frac{c_{XY}}{\sigma_X\sigma_Y} = \frac{E\{xy^*\} - m_X m_Y^*}{\sigma_X\sigma_Y}, \quad \lvert\rho_{XY}\rvert \le 1$

Gaussian distribution	$f_X(\alpha)=\frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{(\alpha-m_X)^2}{2\sigma_X^2}\right)$ $E\left\{(x-m_X)^K\right\}=\begin{cases}1\times3\times5\times\ldots\times(K-1)\sigma_X^K & , \; K \text{ even}\\ 0 & , \; K \text{ odd}\end{cases}$ $f_X(\mathbf{x})=\frac{1}{\left(\sqrt{2\pi}\right)^N\sqrt{\lvert\mathbf{C}_X\rvert}}\exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m}_X)^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{m}_X)\right)$ $\mathbf{x}=[x_1, x_2, \ldots, x_N]^T \quad , \quad \mathbf{m}_X=[m_1, m_2, \ldots, m_N]^T$ $c_{i,j}=E\{(x_i-m_i)(x_j-m_i)\}$
sample mean, sample variance	$\hat{m}_X=\frac{1}{N}\sum_{n=1}^{N}x_n$ $\sigma_X^2=\frac{1}{N}\sum_{n=1}^{N}(x_n-\hat{m}_X)^2 \quad , \quad \sigma_X^2=\frac{1}{N-1}\sum_{n=1}^{N}(x_n-\hat{m}_X)^2$
sum of M statistically independent random variables	$y=\sum_{K=1}^{M}c_K x_K$ $m_Y=\sum_{K=1}^{M}c_K m_{XK} \quad , \quad \sigma_Y^2=\sum_{K=1}^{M}\lvert c_K\rvert^2\sigma_{XK}^2$ $f_Y(\alpha)=\frac{1}{\lvert c_1\rvert}f_{X1}\left(\frac{\alpha}{c_1}\right)*\frac{1}{\lvert c_2\rvert}f_{X2}\left(\frac{\alpha}{c_2}\right)*\ldots*\frac{1}{\lvert c_M\rvert}f_{XM}\left(\frac{\alpha}{c_M}\right)$
discrete-time random processes	$m_X[n]=E\{x[n]\} \quad , \quad \sigma_X^2[n]=E\left\{(x[n]-m_X[n])^2\right\}$ $r_{XY}[k,\ell]=E\{x[k]y^*[\ell]\}$, orthogonal if $r_{XY}[k,\ell]=0$ $c_{XY}[k,\ell]=E\left\{(x[k]-m_X[k])(y[\ell]-m_Y[\ell])^*\right\}$ $c_{XY}[k,\ell]=r_{XY}[k,\ell]-m_X[k]m_Y^*[\ell]$, uncorrelated if $c_{XY}[k,\ell]=0$

WSS discrete-time random processes	$m_X[n] = m_X \quad , \quad \sigma_X^2[n] = \sigma_X^2$ $r_{XY}[k,\ell] = r_{XY}[k-\ell] \quad , \quad r_{XY}[\ell] = r_{YX}^*[-\ell]$
autocorrelation and covariance matrices	$\mathbf{R}_X = E\{\mathbf{x}\mathbf{x}^H\} \quad , \quad \mathbf{C}_X = \mathbf{R}_X - \mathbf{m}_X\mathbf{m}_X^H$ $\mathbf{R}_X$ is Hermitian, Toeplitz and nonnegative definite
eigenvalues and eigenvectors of $\mathbf{R}_X$	$\mathbf{R}_X\mathbf{q} = \lambda\mathbf{q} \quad , \quad \det(\mathbf{R}_X - \lambda\mathbf{I}) = 0$ $\mathbf{q}_j^H\mathbf{R}_X\mathbf{q}_j = \lambda_j\mathbf{q}_j^H\mathbf{q}_j \geq 0 \quad , \quad \lambda_i \neq \lambda_j \quad \Rightarrow \quad \mathbf{q}_i^H\mathbf{q}_j = 0$ if eigenvectors form an orthonormal basis, $\mathbf{Q}$ is unitary and $\mathbf{\Lambda} = \mathbf{Q}^H\mathbf{R}_X\mathbf{Q} \quad , \quad \mathbf{R}_X = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H = \sum_k \lambda_k\mathbf{q}_k\mathbf{q}_k^H$
linear transformation of random vectors	$\mathbf{y} = \mathbf{A}\mathbf{x} \ , \ \mathbf{m}_Y = \mathbf{A}\mathbf{m}_X \ , \ \mathbf{R}_Y = \mathbf{A}\mathbf{R}_X\mathbf{A}^H \ , \ \mathbf{C}_Y = \mathbf{A}\mathbf{C}_X\mathbf{A}^H$ $\mathbf{R}_{XY} = \mathbf{R}_X\mathbf{A}^H \quad , \quad \mathbf{R}_{YX} = \mathbf{A}\mathbf{R}_X \quad , \quad \mathbf{C}_{XY} = \mathbf{C}_X\mathbf{A}^H$
eigenanalysis, KLT	$\mathbf{w} = \mathbf{A}\mathbf{x} \quad , \quad \mathbf{\Lambda}_X = \mathbf{Q}_X^H\mathbf{R}_X\mathbf{Q}_X \quad , \quad \mathbf{C}_W = \mathbf{A}\mathbf{C}_X\mathbf{A}^H$
ergodicity	$E\{\langle x[n]\rangle_N\} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x[n] = m_X$ $E\{\langle r_X[\ell]\rangle_N\} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x[n]x^*[n-\ell] = r_X[\ell]$
white noise	$r_V[\ell] = \sigma_V^2\delta[\ell]$
power spectrum	$P_X(z) = \sum_{\ell=-\infty}^{\infty} r_X[\ell]z^{-\ell} \quad , \quad r_X[\ell] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_X(e^{j\omega})e^{j\omega\ell}d\omega$

filtering random processes	$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$ $m_Y = H(1)m_X \quad , \quad r_{YX}[\ell] = h[\ell] * r_X[\ell]$ $r_Y[\ell] = r_X[\ell] * h[\ell] * h^*[-\ell] = r_X[\ell] * r_H[\ell]$ $\sigma_Y^2 = \mathbf{h}^H \mathbf{R}_X \mathbf{h} = \sum_m \sum_k h^*[m] r_X[m-k] h[k]$ $P_Y(z) = P_X(z)H(z)H^*(1/z^*)$
spectral factorization	if $x[n] = v[n] * h[n]$ and $P_V(z) = \sigma_V^2$, $P_X(z) = \sigma_V^2 Q(z)Q^*(1/z^*)$, $Q(z)$ is minimum phase
ARMA processes and Yule-Walker equations	$H(z) = \dfrac{\sum_{k=0}^{q} b[k]z^{-k}}{1+\sum_{k=1}^{p} a[k]z^{-k}} = \dfrac{B(z)}{A(z)}$ $P_X(z) = \sigma_V^2 \dfrac{B(z)B^*(1/z^*)}{A(z)A^*(1/z^*)}$ $r_x[\ell] + \sum_{k=1}^{p} a[k]r_x[\ell-k] = \begin{cases} \sigma_V^2 c[\ell] & , 0 \le \ell \le q \\ 0 & , \ell > q \end{cases}$ $c[\ell] = \sum_{k=\ell}^{q} b[k]h^*[k-\ell] = \sum_{k=0}^{q-\ell} b[k+\ell]h^*[k]$
AR processes and Yule-Walker equations	$H(z) = \dfrac{b[0]}{1+\sum_{k=1}^{p} a[k]z^{-k}} = \dfrac{b[0]}{A(z)}$ $P_X(z) = \sigma_V^2 \dfrac{\lvert b[0]\rvert^2}{A(z)A^*(1/z^*)}$ $r_x[\ell] + \sum_{k=1}^{p} a[k]r_x[\ell-k] = \sigma_V^2 \lvert b[0]\rvert^2 \delta[\ell] \quad , \quad \ell \ge 0$

MA processes and Yule-Walker equations	$H(z)=\sum_{k=0}^{q} b[k]z^{-k} = B(z)$ $P_X(z)=\sigma_V^2 B(z)B^*(1/z^*)$ $r_x[\ell] = \sigma_V^2 \sum_{k=\ell}^{q} b[k]b^*[k-\ell] = \sigma_V^2 b[\ell] * b^*[-\ell], \quad 0 \le \ell \le q$
Spectrum estimation (periodogram, modified periodogram)	$x_W[n]=w[n]x[n], \qquad w[n]=0,\ n<0 \vee n>N-1$ $\hat{r}_X[k]=\frac{1}{N}x_W[k]*x_W^*[-k] \overset{F}{\longleftrightarrow} \hat{P}_{PER}\left(e^{j\omega}\right)$ $\hat{P}_{PER}\left(e^{j\omega}\right)=\frac{1}{N}\left\|X_W\left(e^{j\omega}\right)\right\|^2 = \frac{1}{2\pi N}P_X\left(e^{j\omega}\right)*\left\|W\left(e^{j\omega}\right)\right\|^2$ $\hat{P}_{PER}\left(e^{j\omega}\right)=\sum_{k=1-N}^{N-1}\hat{r}_W[k]e^{-jk\omega} \rightarrow \hat{P}_{PER}\left(e^{j\omega_k}\right), \quad \omega_k = k\frac{2\pi}{N}$ $x[n] \rightarrow x_W[n]=w[n]x[n] \xrightarrow{DFT} X_W[k]$ $X_W[k] \rightarrow \frac{1}{N}\left\|X_W[k]\right\|^2 = \hat{P}_{PER}\left(e^{j\omega_k}\right), \quad k=0,1,\ldots N-1$

arithmetic progression	$\sum_{k=1}^{M} k = \frac{M(M+1)}{2}$
geometric progression	$\sum_{k=0}^{M} \alpha^k = \frac{1-\alpha^{M+1}}{1-\alpha}$
Sum of squares	$\sum_{k=1}^{M} k^2 = \frac{M(M+1)(2M+1)}{6}$