Summary

- *Frequency-domain representation of discrete signals and systems*
  - *Response of an LTI system to a complex exponential*
  - *Fourier representation of a discrete-time sequence*
- *A Review of the discrete-time Fourier Transform (DTFT)*
  - *Symmetry properties of the Fourier Transform*
  - *Theorems regarding the Fourier Transform*
  - *Table of Fourier pairs*
- *The DTFT of the auto-correlation and of the cross-correlation*
  - *the DTFT of the auto-correlation*
  - *the DTFT of the cross-correlation*
  - *examples*

Frequency-domain representation of discrete signals & systems

- **Question:** what is the output of an LTI system when the input is a complex exponential ? $x[n] = e^{j\omega n} \quad , \quad -\infty < n < +\infty$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n]h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]e^{j\omega(n-k)} = \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}e^{j\omega n} = H\left(e^{j\omega}\right)e^{j\omega n}$$

  - **Answer**: it's the complex exponential possibly modified in magnitude and phase according to the frequency response of the LTI system.

  - **Note**: this result reveals that $e^{j\omega n}$ is an eigen function of the LTI system and that $H(e^{j\omega})$ is the eigen value of the system at the angular frequency ω radians.

- **Definition of the frequency response of an LTI system** (obtained by computing the Fourier transform of its impulse response)

$$H\left(e^{j\omega}\right) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} = \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)}$$

  - $|H(e^{j\omega})|$ $\rightarrow$ absolute value of the frequency response of the system
  - $\angle H(e^{j\omega})$ $\rightarrow$ phase of the frequency response of the system

Frequency-domain representation of discrete signals & systems

– **Example**: what is the response of an LTI system, with h[n] real, to the input $x[n]=A\cos(\omega_0 n+\phi)$?

– **Answer**: x[n] may be expressed in a convenient way: $x[n]=\frac{A}{2}\left[e^{j(\omega_0 n+\phi)}+e^{-j(\omega_0 n+\phi)}\right]$

and then:

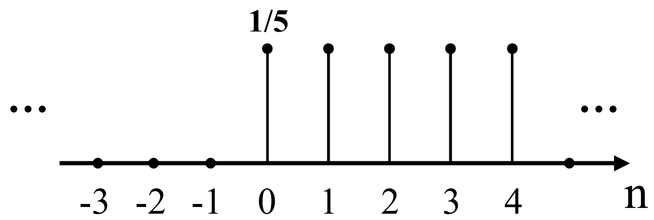
$$y[n]=\frac{A}{2}\left[H\left(e^{j\omega_0}\right)e^{j(\omega_0 n+\phi)}+H\left(e^{-j\omega_0}\right)e^{-j(\omega_0 n+\phi)}\right]=A\left|H\left(e^{j\omega_0}\right)\right|\cos\left[\omega_0 n+\phi+\angle H\left(e^{j\omega_0}\right)\right]$$

– Important property of $H(e^{j\omega})$

given the <u>periodicity</u> of the discrete complex exponential, $e^{j\omega n}$, the frequency response $H(e^{j\omega})$ is periodic with period 2π, so that in order to characterize it completely, it is sufficient to represent the magnitude and phase considering a frequency span of 2π radians, e.g., between $-\pi$ and $+\pi$ or 0 and 2π.

– Example: what is the frequency response of a moving-average filter of length 5 ?

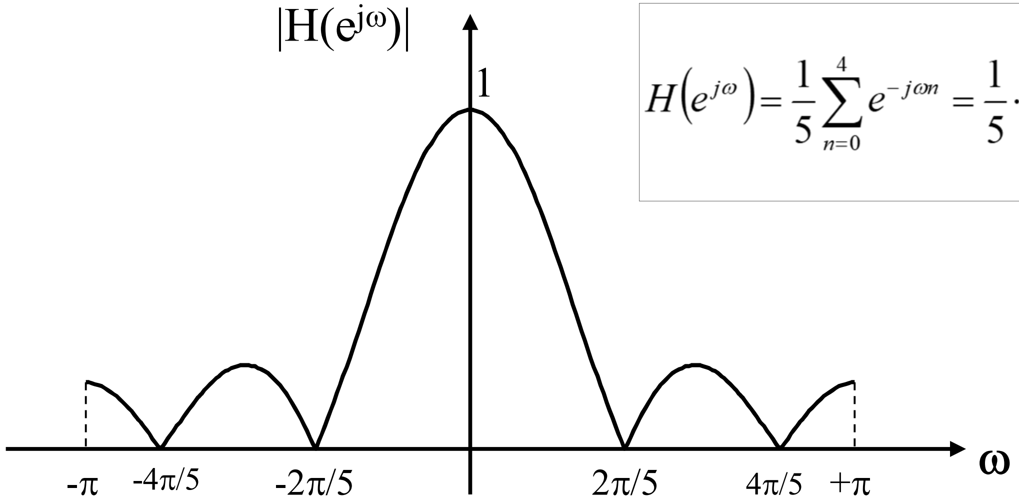
$$h[n]=\begin{cases}1/5 & 0\le n\le 4\\ 0 & outros\end{cases}$$



Frequency-domain representation of discrete signals & systems

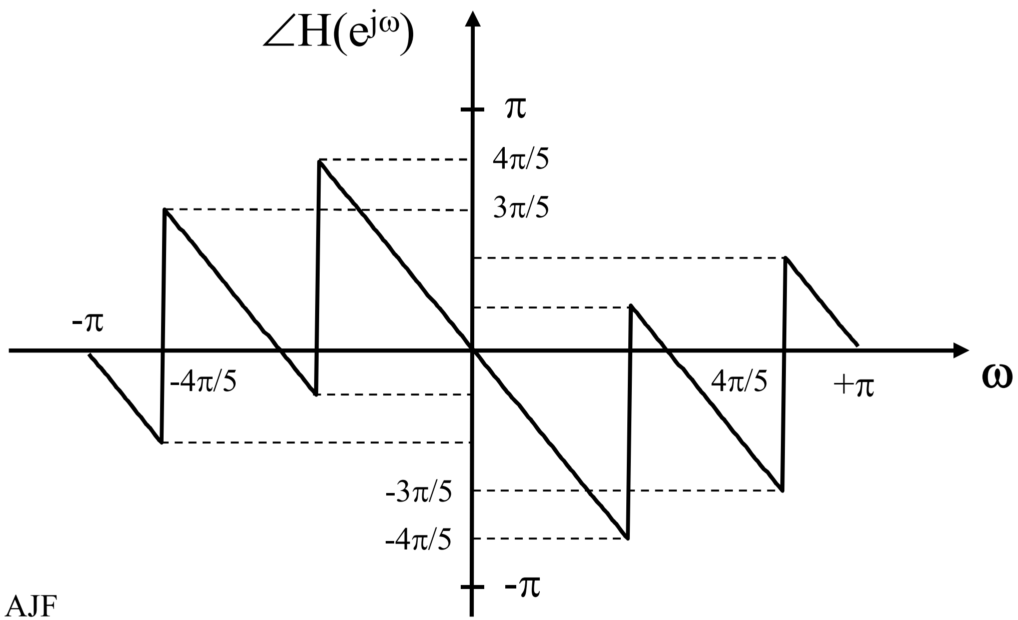
– **Answer**: using the definition of the time-discrete Fourier transform:

$$H\left(e^{j\omega}\right)=\frac{1}{5}\sum_{n=0}^{4}e^{-j\omega n}=\frac{1}{5}\cdot\frac{1-e^{-j5\omega}}{1-e^{-j\omega}}=\frac{1}{5}e^{-j2\omega}\frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}}=\left|H\left(e^{j\omega}\right)\right|e^{j\angle H\left(e^{j\omega}\right)}$$



NOTE 1: the magnitude function is even.

NOTE 2: the phase function is odd.



Question 1: why is that $\angle H(e^{j\omega}) \neq -2\omega$?

(note that $-1=e^{\pm j\pi}$)

Question 2: why is that in this representation of $\angle H(e^{j\omega})$ we say that the phase is *wrapped* ?

(what is the fundamental period in the representation of phase ?)

Fourier representation of a discrete sequence

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega \quad \overset{F}{\longleftrightarrow} \quad X\left(e^{j\omega}\right)=\left|X\left(e^{j\omega}\right)\right|e^{j\angle X\left(e^{j\omega}\right)}=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}$$

- the Fourier transform of a discrete-time signal x[n] is periodic with period 2π and exists if x[n] is absolutely summable
- the inverse Fourier transform allows to synthesize x[n] using a period of its representation in the frequency domain

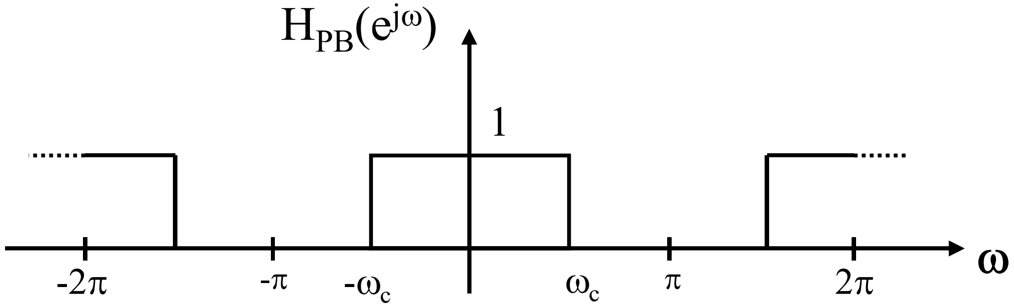
– **Example:**

$$x[n]=a^n u[n] \quad \overset{F}{\longleftrightarrow} \quad X\left(e^{j\omega}\right)=\sum_{n=0}^{+\infty}a^n e^{-j\omega n}=\sum_{n=0}^{+\infty}\left(ae^{-j\omega}\right)^n=\frac{1}{1-ae^{-j\omega}}$$

if $|ae^{-j\omega}| < 1 \quad \therefore \quad |a| < 1$

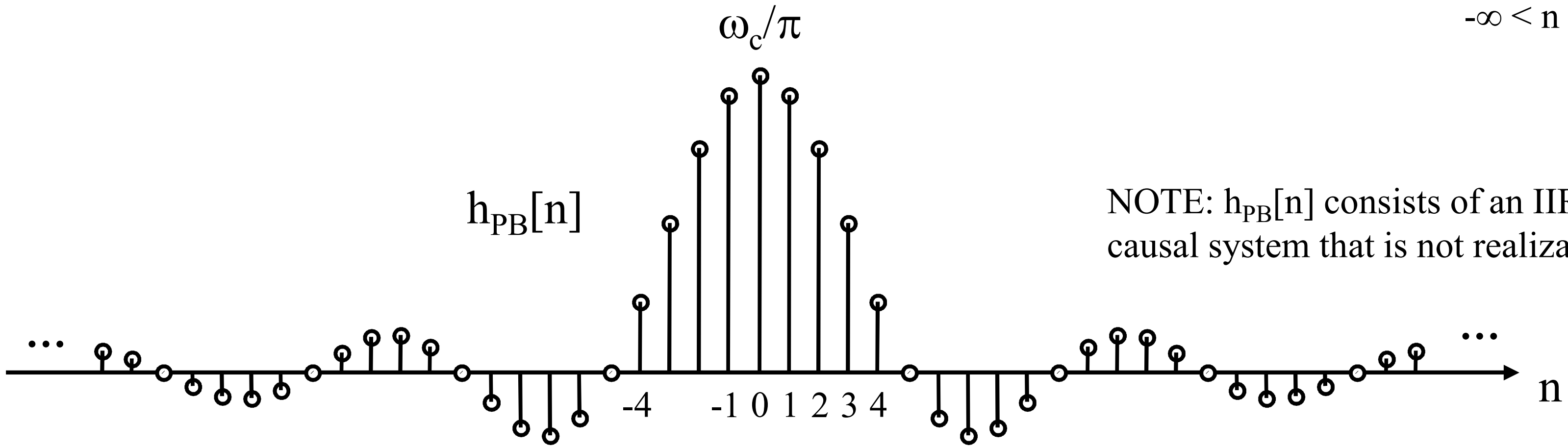
Fourier representation of a discrete sequence

- **Example:** what is the impulse response of an ideal low-pass filter ?



$$H_{PB}\left(e^{j\omega}\right) \xleftrightarrow{\;\;F\;\;} h_{PB}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{PB}\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin n\omega_c}{n\pi}$$

$-\infty < n < +\infty$



NOTE: $h_{PB}[n]$ consists of an IIR non-causal system that is not realizable !

NOTE+: the response $h_{PB}[n]$ is not absolutely summable, but its square is summable, which highlights the fact that a filter resulting fom $h_{PB}[n]$ by limiting its length, is the best approximation, in the mean-square sense, to $H_{PB}(e^{j\omega})$ (*i.e.* to the ideal filter).

Fourier representation of a discrete sequence

– special cases

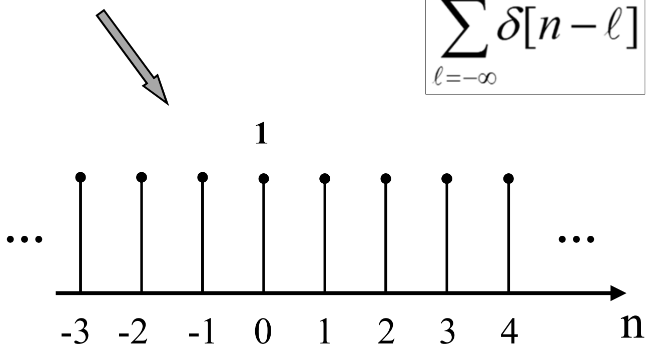
these are special cases because they are neither absolutely summable nor square-summable, they arise from the theory of generalized functions but they are very important in the analysis of signals and discrete-time systems:

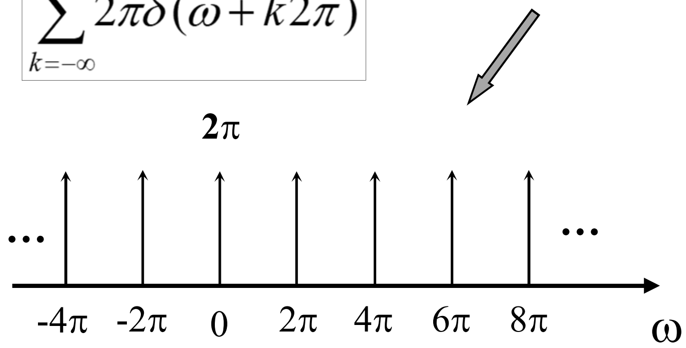
- train of impulses

unitary impulses

$$\sum_{\ell=-\infty}^{+\infty} \delta[n-\ell] \quad \overset{\mathsf{F}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega + k2\pi)$$

Dirac impulses





- unitary complex exponential

$$e^{j\omega_0 n} \quad \overset{\mathsf{F}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 + k2\pi)$$

- unitary step

$$u[n] \quad \overset{\mathsf{F}}{\longleftrightarrow} \quad \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega + k2\pi)$$

Symmetry properties of the time-discrete Fourier transform

– given x[n] , we may express $x[n]=x_e[n]+x_o[n]$ where:

$$x_e[n]=\frac{1}{2}\left(x[n]+x^*[-n]\right)=x_e^*[-n]$$

- $x_e[n]$ is the <u>conjugate symmetric sequence</u> of x[n]; in case x[n] is real, $x_e[n]$ is also known as the *even* component of x[n] since $x_e[n]=x_e[-n]$

$$x_o[n]=\frac{1}{2}\left(x[n]-x^*[-n]\right)=-x_o^*[-n]$$

- $x_o[n]$ is the <u>conjugate anti-symmetric sequence</u> of x[n]; in case x[n] is real, $x_o[n]$ is also known as the *odd* component of x[n] since $x_o[n]=-x_o[-n]$

– similarly, $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$

$$X_e\left(e^{j\omega}\right)=\frac{1}{2}\left[X\left(e^{j\omega}\right)+X^*\left(e^{-j\omega}\right)\right]=X_e^*\left(e^{-j\omega}\right)$$

- $X_e(e^{j\omega})$ is the <u>conjugate symmetric function</u> of $X(e^{j\omega})$, $X_e(e^{j\omega})$ is also said the *even* component of $X(e^{j\omega})$ when $X(e^{j\omega})$ is real-valued

$$X_o\left(e^{j\omega}\right)=\frac{1}{2}\left[X\left(e^{j\omega}\right)-X^*\left(e^{-j\omega}\right)\right]=-X_o^*\left(e^{-j\omega}\right)$$

- $X_o(e^{j\omega})$ is the <u>conjugate anti-symmetric function</u> of $X(e^{j\omega})$, $X_o(e^{j\omega})$ is also said the *odd* component of $X(e^{j\omega})$ when $X(e^{j\omega})$ is real-valued

Main symmetry properties of the time-discrete Fourier transform

$$x[n] \text{ (complex)} \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

$x^*[n]$	$X^*\left(e^{-j\omega}\right)$
$x^*[-n]$	$X^*\left(e^{j\omega}\right)$
$\Re\{x[n]\}$	$X_e\left(e^{j\omega}\right)$ conjugate symmetric part of $X(e^{j\omega})$
$j\Im\{x[n]\}$	$X_o\left(e^{j\omega}\right)$ conjugate anti-symmetric part of $X(e^{j\omega})$
$x_e[n]$	$\Re\left\{X\left(e^{j\omega}\right)\right\}$
$x_o[n]$	$j\Im\left\{X\left(e^{j\omega}\right)\right\}$

$$x[n] \text{ (real-valued)} \overset{F}{\longleftrightarrow} X(e^{j\omega})=X_{\Re}(e^{j\omega})+jX_{\Im}(e^{j\omega})=X^*(e^{-j\omega})$$

i.e. the transform is conjugate symmetric :

$$X_{\Re}\left(e^{j\omega}\right)=X_{\Re}\left(e^{-j\omega}\right)$$

$$X_{\Im}\left(e^{j\omega}\right)=-X_{\Im}\left(e^{-j\omega}\right)$$

$$\left|X\left(e^{j\omega}\right)\right|=\left|X\left(e^{-j\omega}\right)\right|$$

$$\angle X\left(e^{j\omega}\right)=-\angle X\left(e^{-j\omega}\right)$$

$x_e[n]$	$X_{\Re}\left(e^{j\omega}\right)$
$x_o[n]$	$jX_{\Im}\left(e^{j\omega}\right)$

Review of the main Fourier transform theorems

(relate operations involving discrete sequences and the corresponding operations in the Fourier domain)

$$x[n],\ y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}),\ Y(e^{j\omega})$$

linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$	
shift in n	$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	n_d inteiro
shift in ω	$e^{j\omega_0 n}x[n]$	$X\left[e^{j(\omega-\omega_0)}\right]$	
'time' reversal	$x[-n]$	$X(e^{-j\omega})$	
differentiation in ω	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$	
convolution	$x[n]*y[n]$	$X(e^{j\omega})\cdot Y(e^{j\omega})$	
product	$x[n]\cdot y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	(periodic convolution)

why is there no "differentiation" in n ?

Parseval theorem

$$\sum_{n=-\infty}^{+\infty}x[n]\cdot y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Parseval theorem (particular case)

$$\underbrace{E=\sum_{n=-\infty}^{+\infty}|x[n]|^2}_{\text{energy}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left|X(e^{j\omega})\right|^2}_{\text{energy spectral density}}d\omega$$

Tabela de pares de Fourier

example: $a^n u[n], \quad |a|<1$ $\quad$ x[n] $\overset{F}{\longleftrightarrow}$ X(e$^{j\omega}$) $\quad \dfrac{1}{1-ae^{-j\omega}}$

x[n]	X(e$^{j\omega}$)
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$\sum_{\ell=-\infty}^{+\infty}\delta[n-\ell]$	$\sum_{k=-\infty}^{+\infty}2\pi\delta(\omega+k2\pi)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{+\infty}2\pi\delta(\omega-\omega_0+k2\pi)$
$u[n]$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega+k2\pi)$
$(n+1)a^n u[n], \quad \|a\|<1$	$1/\|1-ae^{-j\omega}\|^2$
$\begin{cases}1, & 0\le n\le M\\ 0, & outros\end{cases}$	$\dfrac{\sin(M+1)\dfrac{\omega}{2}}{\sin\dfrac{\omega}{2}}\cdot e^{-j\omega\frac{M}{2}}$
$\cos(\omega_0 n+\phi)$	$\pi\sum_{k=-\infty}^{+\infty}\left[e^{j\phi}\delta(\omega-\omega_0+k2\pi)+e^{-j\phi}\delta(\omega+\omega_0+k2\pi)\right]$
$\dfrac{\sin n\omega_c}{n\pi}$	$\begin{cases}1, & \|\omega\|<\omega_c\\ 0, & \omega_c<\|\omega\|\le\pi\end{cases}$
$r^n\dfrac{\sin\omega_P(n+1)}{\sin\omega_P}u[n], \quad \|r\|<1$	$1/\left(1-2r\cos\omega_P e^{-j\omega}+r^2e^{-j2\omega}\right)$

Question: what is a practical way to find the inverse Fourier transform ?

- **Example:** $X(e^{j\omega}) = \dfrac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})}$ **, causal** $\xleftrightarrow{\;F\;}$ x[n]= ?

if M<N and poles are first-order, then: $$X(e^{j\omega}) = \frac{\prod_{\ell=1}^{M}(1-c_\ell e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})} = \sum_{k=1}^{N} \frac{A_k}{1-d_k e^{-j\omega}}$$

with : $A_k = (1-d_k e^{-j\omega})X(e^{j\omega})\Big|_{e^{j\omega}=d_k}$

and thus: $$\frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} = \frac{a/(a-b)}{1-ae^{-j\omega}} + \frac{b/(b-a)}{1-be^{-j\omega}}$$

which leads to: $$x(n) = \frac{a}{a-b}a^n u[n] + \frac{b}{b-a}b^n u[n]$$

<u>Not to forget !</u>

Fundamentals of Signal Processing, week 2
FEUP-DEEC, September 19-23, 2022

- the DTFT of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, x^*[k-\ell]$$

considering the DTFT properties

$$x[\ell] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$
$$x^*[\ell] \overset{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{-j\omega}\right)$$
$$x[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{-j\omega}\right)$$
$$x^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{j\omega}\right)$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} R_x\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) \cdot X^*\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

Where $R_x\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$ is called the spectral density of energy

- the DTFT of the auto-correlation (cont.)
  - the Wiener-Khinchine Theorem: the auto-correlation and the spectral density of energy form a Fourier pair

$$r_x[\ell] \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad R_x\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

thus,

$$r_x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R\left(e^{j\omega}\right) e^{j\omega\ell} d\omega$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} R\left(e^{j\omega}\right) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

which reflects the Parseval Theorem

- the DTFT of the cross-correlation

the cross-correlation is defined as (we admit energy signals)

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\, y^*[k-\ell]$$

considering the DTFT properties

$$x[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(e^{j\omega})$$

$$y[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(e^{j\omega})$$

$$y^*[\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y^*(e^{-j\omega})$$

$$y[-\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(e^{-j\omega})$$

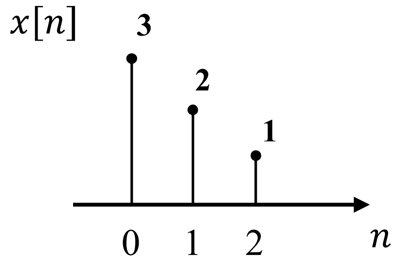
$$y^*[-\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y^*(e^{j\omega})$$

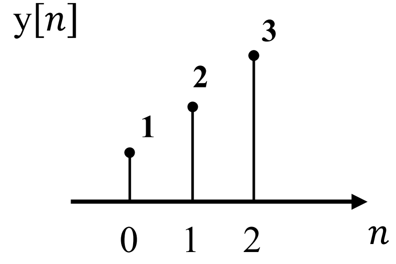
then

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R_{xy}(e^{j\omega}) = X(e^{j\omega}) \cdot Y^*(e^{j\omega})$$

- examples

let us admit two discrete-time signals, $x[n]$ and $\mathrm{y}[n]$





it can be easily concluded that

$$x[\ell] = 3\delta[\ell] + 2\delta[\ell-1] + \delta[\ell-2] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$y[\ell] = \delta[\ell] + 2\delta[\ell-1] + 3\delta[\ell-2] \overset{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\omega}\right) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$R_x\left(e^{j\omega}\right) = 3e^{j2\omega} + 8e^{j\omega} + 14 + 8e^{-j\omega} + 3e^{-j2\omega} = R_y\left(e^{j\omega}\right),\ \text{(why ?)}$$

$$R_{xy}\left(e^{j\omega}\right) = 9e^{j2\omega} + 12e^{j\omega} + 10 + 4e^{-j\omega} + e^{-j2\omega}$$