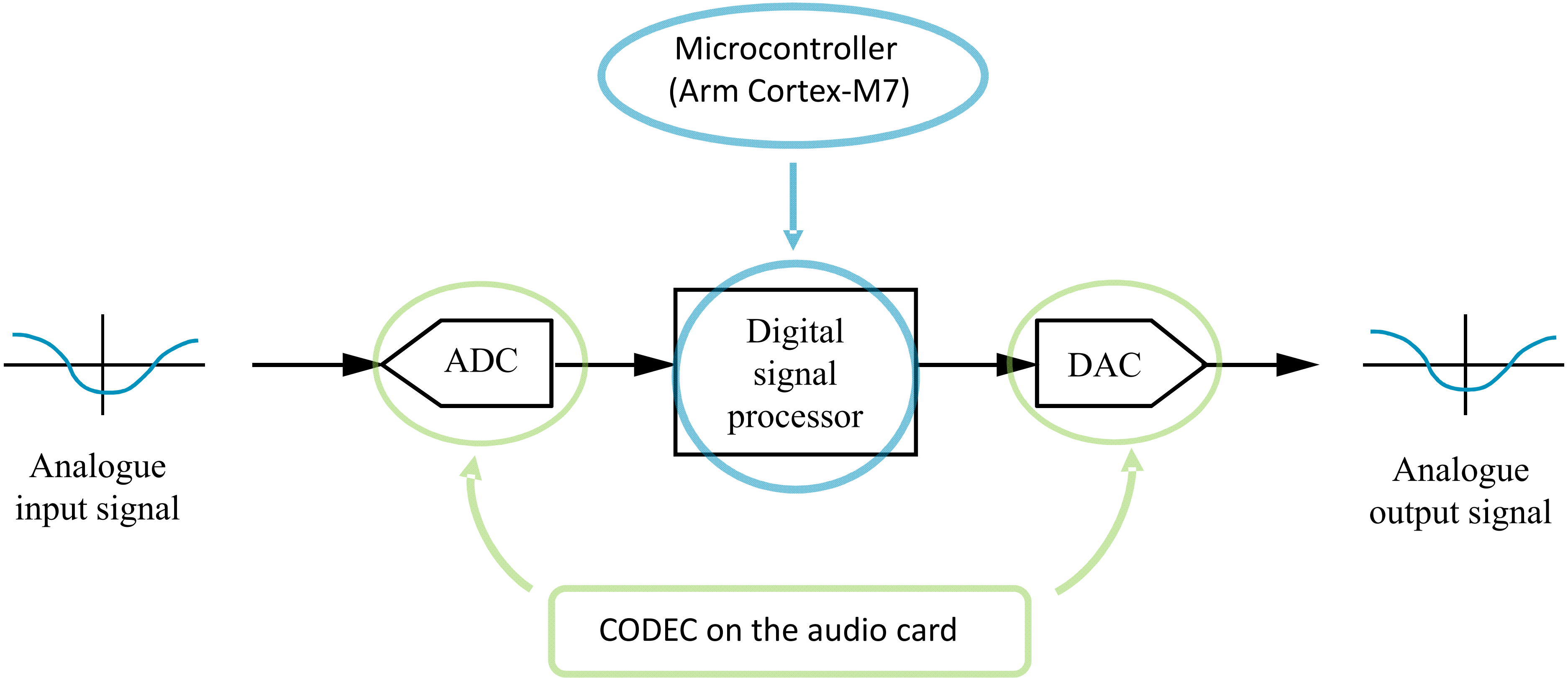
Sampling, Reconstruction, and Aliasing

Review of Complex Exponentials and Fourier Analysis

Digital Signal Processing System



Pre-requisite Theory

- Complex exponentials
- Continuous-time Fourier transform
  - properties
  - important Fourier transform pairs

The Fourier Transform

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Complex, continuous, aperiodic

$$-\infty < t < \infty$$

Forward Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Complex, continuous, aperiodic

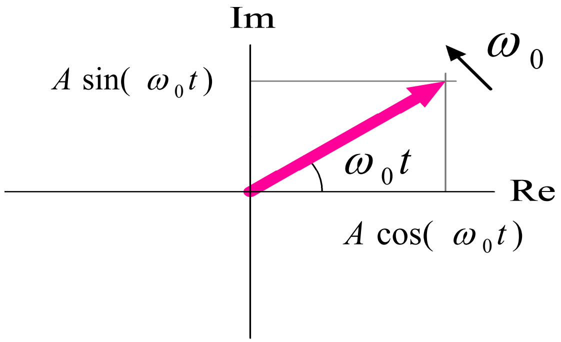
$$-\infty < \omega < \infty$$

Complex exponentials are a fundamental part of Fourier analysis and frequency domain representation of signals.

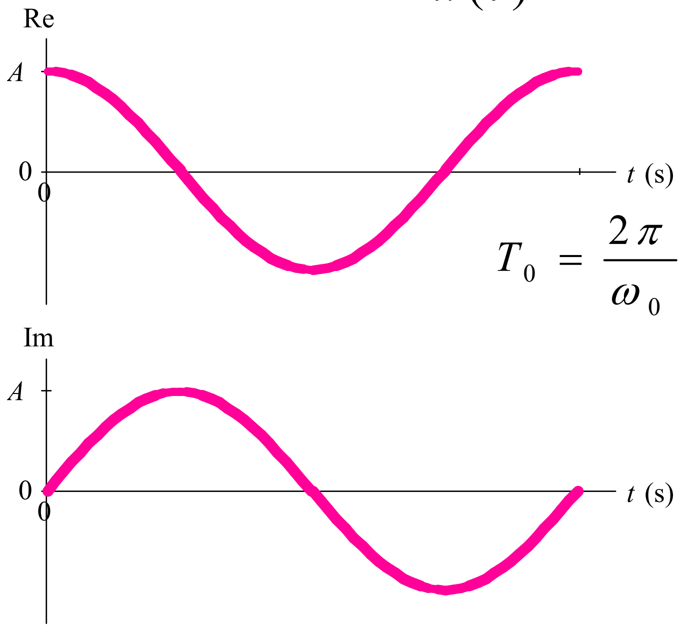
Complex Exponentials

Complex exponential $Ae^{j\omega_0 t}$ represented in three different ways

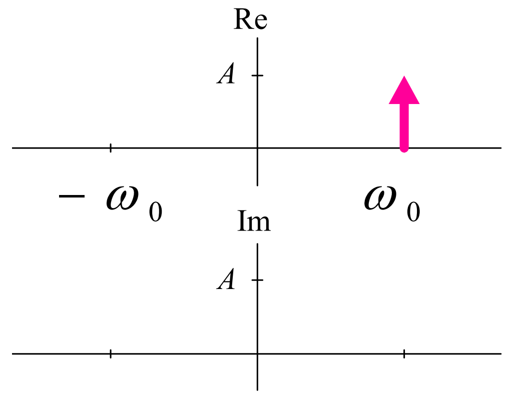
Complex-plane



Time-domain $x(t)$



Frequency-domain $X(\omega)$

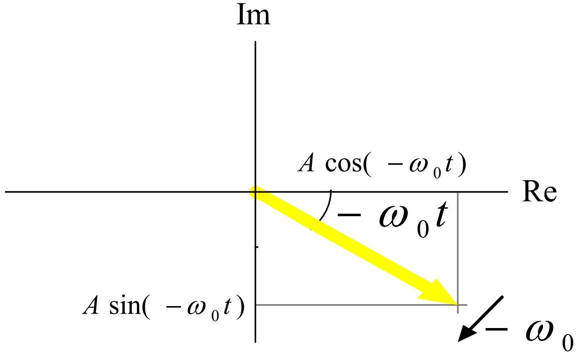


Euler's formula: $Ae^{j\omega t} = A\cos(\omega_0 t) + jA\sin(\omega_0 t)$

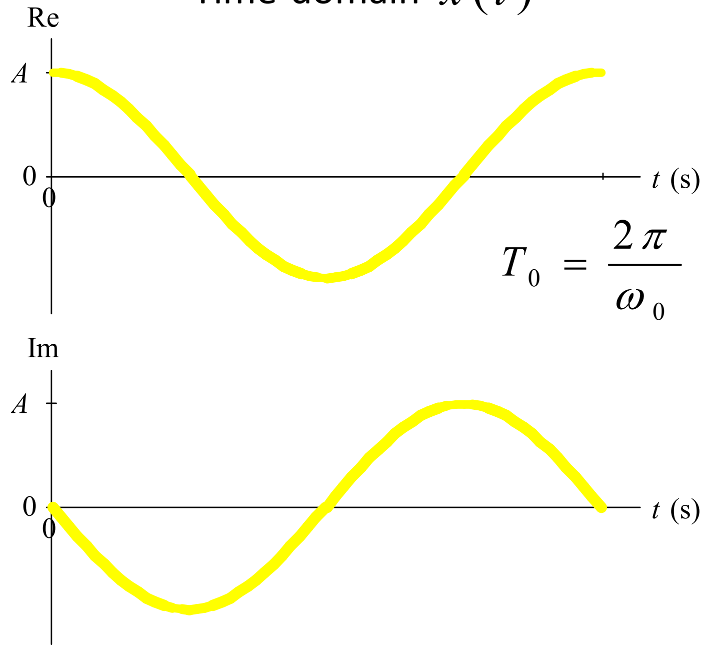
Complex Exponentials

Complex exponential $Ae^{-j\omega_0 t}$ represented in three different ways

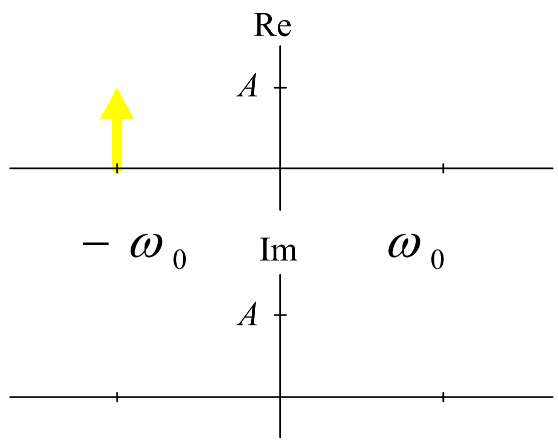
Complex-plane



Time-domain $x(t)$



Frequency-domain $X(\omega)$



Euler's formula: $Ae^{-j\omega t} = A\cos(\omega_0 t) - jA\sin(\omega_0 t)$

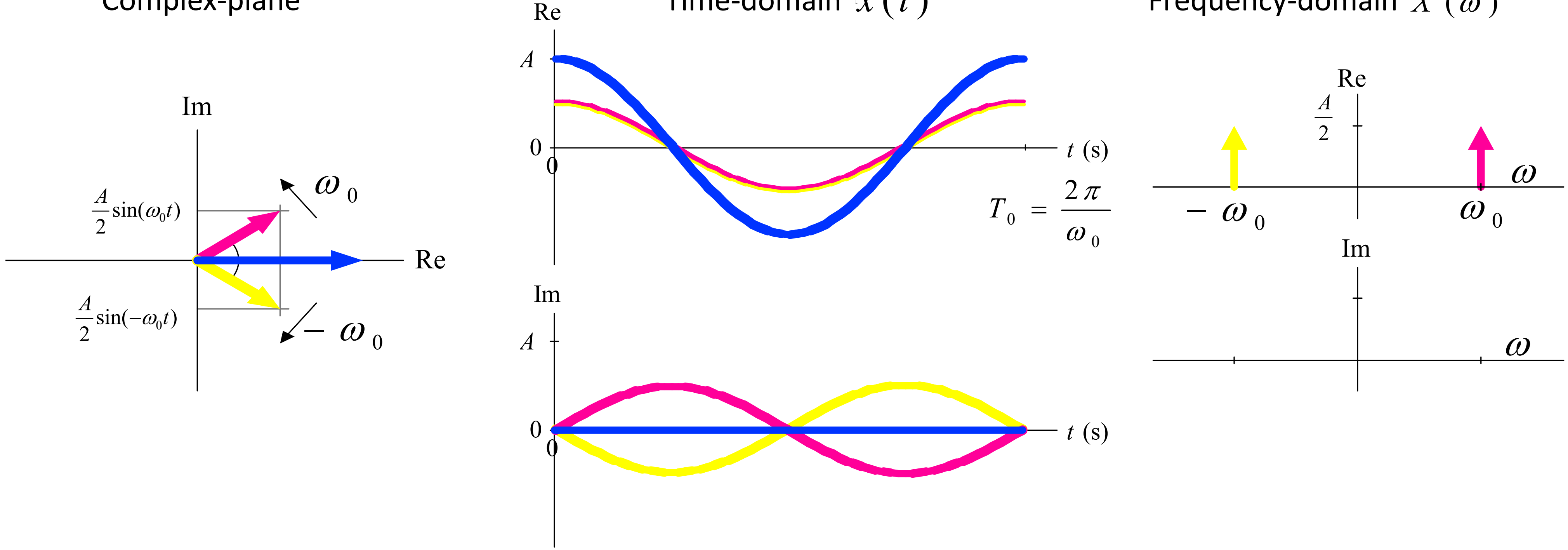
Real-valued Cosine

Real-valued sinusoid $A\cos(\omega_0 t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$ represented in three different ways

Complex-plane

Time-domain $x(t)$

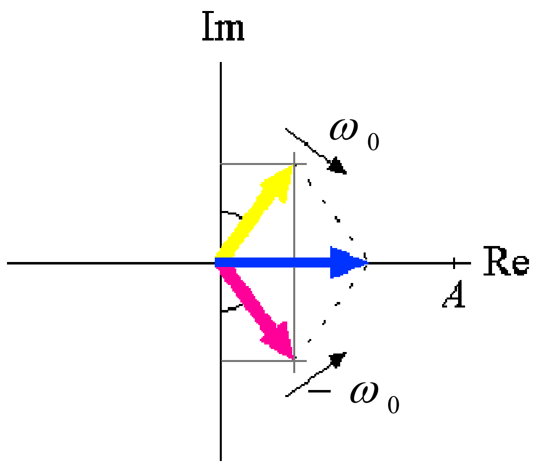
Frequency-domain $X(\omega)$



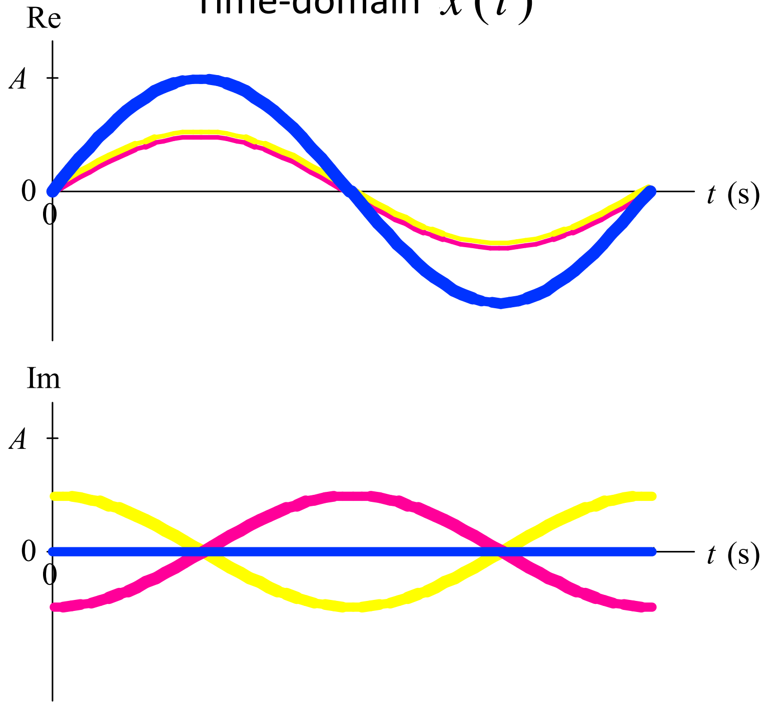
Real-valued Sine

Real-valued sinusoid $A\sin(\omega_0 t) = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$ represented in three different ways

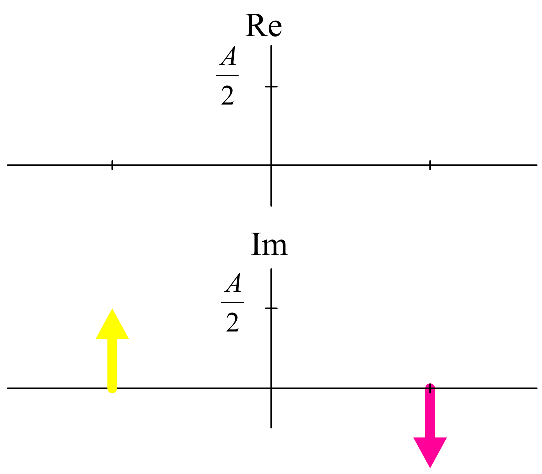
Complex-plane



Time-domain $x(t)$



Frequency-domain $X(\omega)$



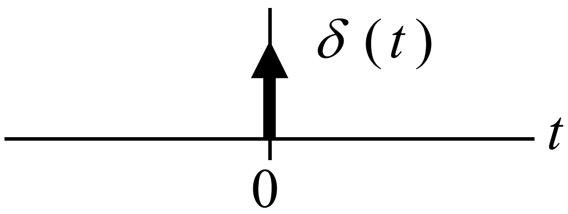
Fourier Transform of an Impulse

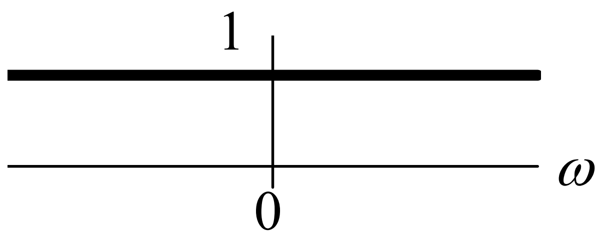
$$x(t) = \delta(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$d(t)$ is zero for $t \neq 0$, and over an infinitesimal interval of time around $t = 0$ (for which the complex exponential term is equal to 1), its integral with respect to time is equal to unity, and hence

$$X(\omega) = 1$$





Fourier Transform of a Constant

The inverse Fourier transform of a constant is equal to an impulse.

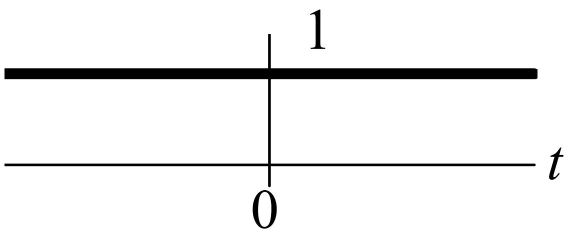
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

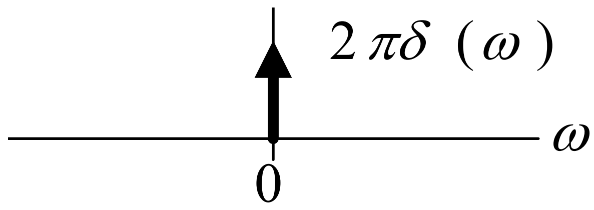
Substitute $-\omega$ for t and t for ω

An impulse in the frequency domain

$$\delta(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jt\omega}\, dt$$

The Fourier transform of unity in the time domain





This is an example of the duality property of the FT.

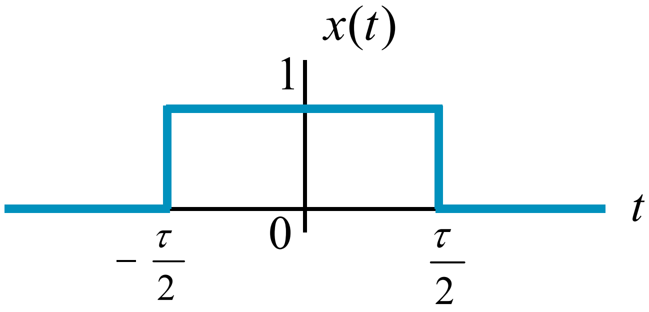
Duality Property

$$\text{If} \quad X(\omega) = F\{x(t)\}$$

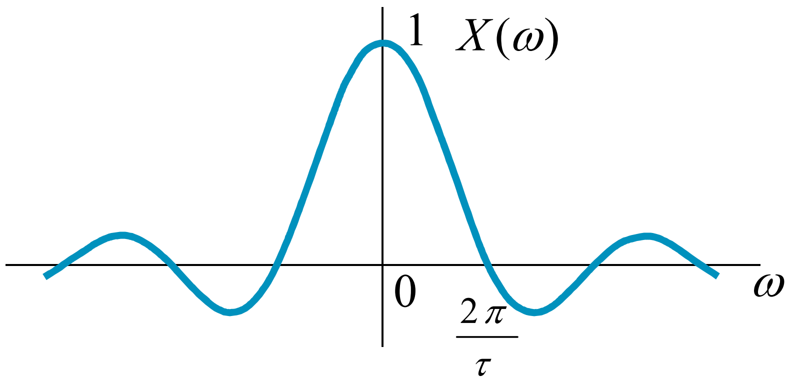
$$\text{then} \quad x(-\omega) = \frac{1}{2\pi} F\{X(t)\}$$

Fourier Transform of a Rectangular Pulse

$$x(t) = \begin{cases} 1, & |t| < \dfrac{\tau}{2} \\ 0, & |t| \ge \dfrac{\tau}{2} \end{cases}$$



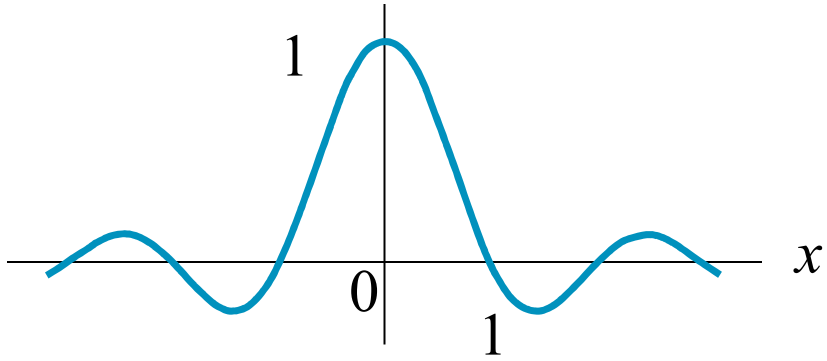
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt = \frac{1}{j\omega}\left[e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}} \right] = \frac{\tau \sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$



sinc Function

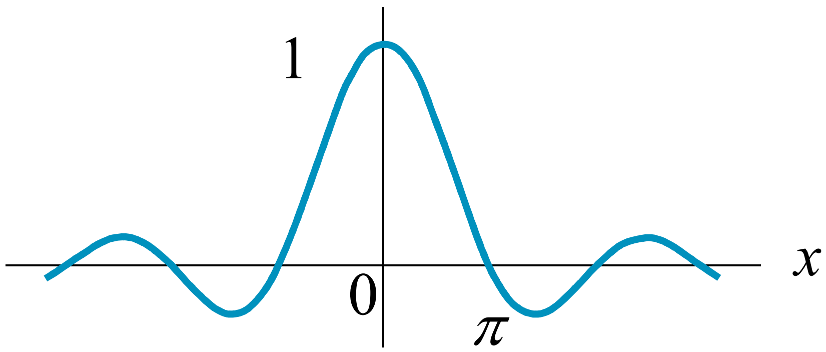
Normalized sinc function

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Un-normalized sinc function

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$



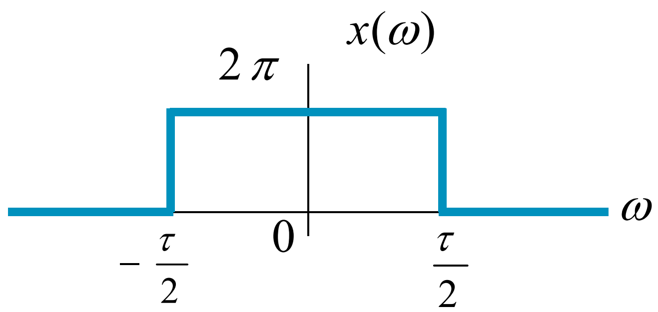
Fourier Transform of a sinc Function

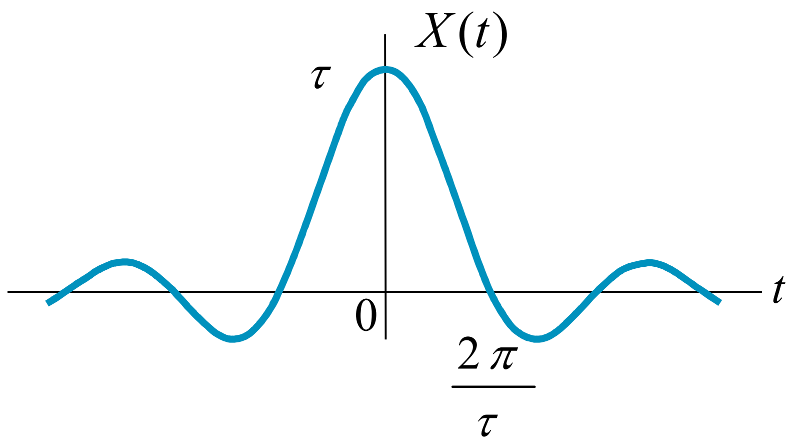
Using the duality property and the previous result,

$$x(-\omega) = \begin{cases} 2\pi, & |-\omega| < \frac{\tau}{2} \\ 0, & |-\omega| \geq \frac{\tau}{2} \end{cases}$$

is the Fourier transform of

$$X(t) = \tau \ \text{sinc}\left(\frac{t\tau}{2\pi}\right)$$

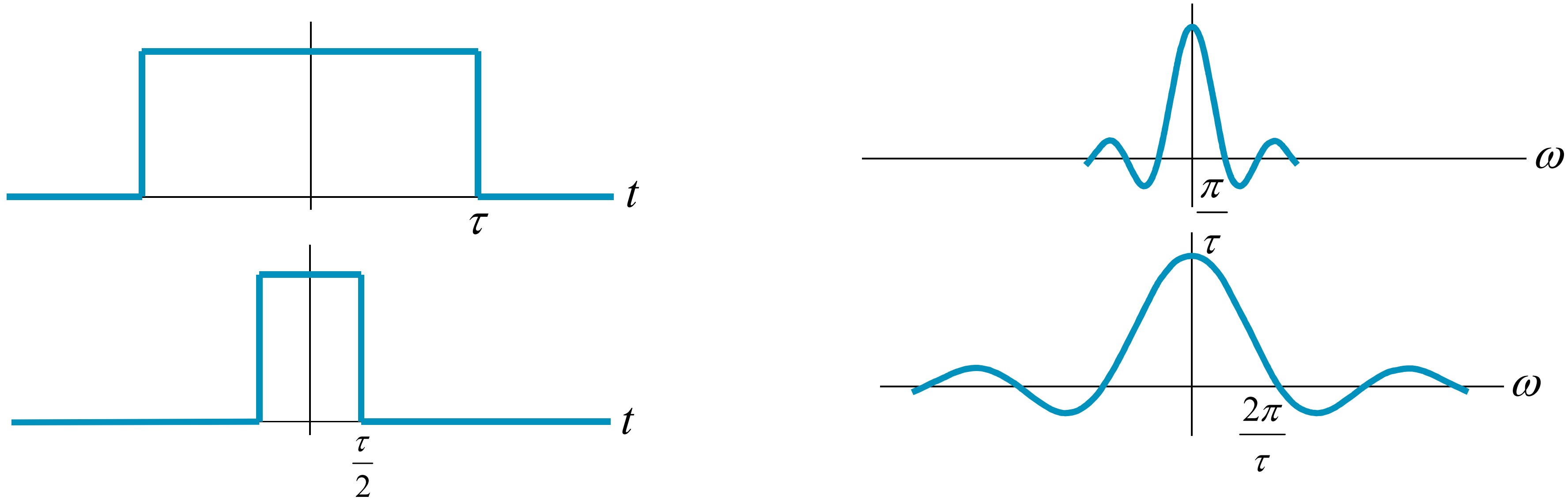




Scaling Property of the Fourier Transform

If $G(\omega) = F\{g(t)\}$

then $\frac{1}{|a|} G\left(\frac{\omega}{a}\right) = F\{g(at)\}$ where a is a real-valued constant



Widening a function in one domain corresponds to narrowing its representation in the other.

Frequency Shifting Property

$$\text{if} \quad G(\omega) = F\{g(t)\}$$

$$\text{then} \quad F\{g(t)e^{j\omega_0 t}\} = G(\omega - \omega_0)$$

Multiplication of a time-domain function by a complex exponential shifts its frequency domain representation along the frequency axis.

Convolution Property

If $G_1(\omega) = F\{g_1(t)\}$

and $G_2(\omega) = F\{g_2(t)\}$

then $F\{g_1(t) * g_2(t)\} = G_1(\omega)G_2(\omega)$

and $F\{2\pi\, g_1(t)g_2(t)\} = G_1(\omega) * G_2(\omega)$

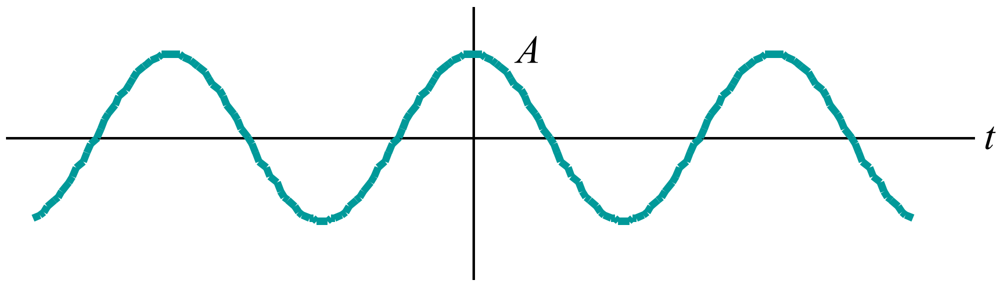
Convolution in the time-domain is equivalent to multiplication in the frequency-domain.

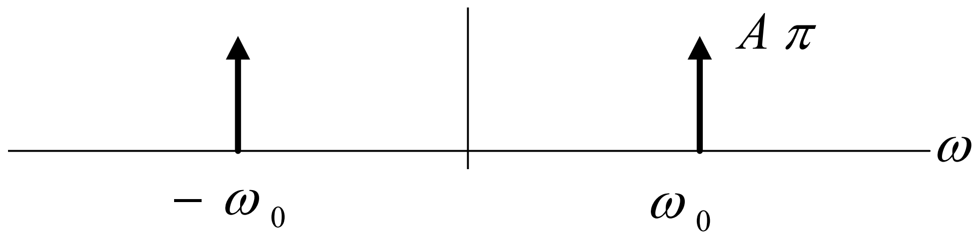
Fourier Transform of a Sinusoid

$$x(t) = A\cos(\omega_0 t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

Use the frequency shifting property $F\left\{g(t)e^{j\omega_0 t}\right\} = G(\omega - \omega_0)$

and the previous result that $F\{1\} = 2\pi\delta(\omega)$





Even and Odd Functions

Even function $$f(x) = f(-x)$$

Odd function $$f(x) = -f(-x)$$

$x(t)$	$X(\omega)$
Real-valued and even	Real-valued and even
Real-valued and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginary and odd	Real-valued and odd

Fourier Transform of an Impulse Train

An infinite train of impulses in the time-domain

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

(sometimes known as the Shah function)
is interesting because (allowing for scaling) its Fourier transform is itself,

$$X(\omega) = \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \qquad \omega_s = \frac{2\pi}{T_s}$$

i.e., an infinite train of impulses in the frequency domain.

The Fourier transform of an infinite train of impulses in the time-domain is an infinite train of impulses in the frequency-domain.