Overview

- *Analysis and representation of LTI systems in the frequency domain*

  – *Basic concepts*
    - *alternative complete characterizations of an LTI system*
    - *ideal frequency-selective filters*
    - *phase distortion and group delay*
    - *difference equation and transfer function*
    - *inverse system*

  – *All-pass systems*

  – *Minimum-phase and maximum-phase systems*

  – *Linear-phase systems*
    - *definition*
    - *FIR systems of type 1, type 2, type 3 and type 4*
    - *Zero location of FIR linear-phase systems*
    - *relation between linear-phase, minimum and maximum phase FIR systems*

- Alternative complete characterizations of an LTI system

  – using the impulse response h[n]: $$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

  – using the frequency response of the system, *i.e.* the Fourier transform of h[n], presuming convergence:

$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega}) = \left|H(e^{j\omega})\right| \cdot \left|X(e^{j\omega})\right| e^{j\left[\angle H(e^{j\omega}) + \angle X(e^{j\omega})\right]}$$

  - where $|H(e^{j\omega})|$ is the magnitude response of the system
  - where $\angle H(e^{j\omega})$ is the phase response of the system

  – using the transfer function of the system, *i.e.* the Z transform of h[n] and the associated region of convergence (admitting it exists):

$$Y(z) = H(z) \cdot X(z)$$

→ these last two alternatives are particularly important in the representation and analysis of discrete-time systems because they reveal many of its properties and characteristics.

- Ideal frequency-selective filters

$\rightarrow$ they amplify or attenuate, as desired, specific frequency regions

– Example 1: ideal low-pass filter

$$H_{PB}\left(e^{j\omega}\right)=\begin{cases}1, & |\omega|<\omega_p\\ 0, & \omega_p<|\omega|\le\pi\end{cases} \quad \overset{F}{\longleftrightarrow} \quad h_{PB}[n]=\frac{\sin\omega_P n}{\pi\, n}\ , \quad -\infty<n<+\infty$$

– Example 2: ideal high-pass filter

$$H_{PA}\left(e^{j\omega}\right)=\begin{cases}0, & |\omega|<\omega_p\\ 1, & \omega_p<|\omega|\le\pi\end{cases} \;\Leftrightarrow\; 1-H_{PB}\left(e^{j\omega}\right) \quad \overset{F}{\longleftrightarrow} \quad h_{PA}[n]=\delta[n]-h_{PB}[n]=\delta[n]-\frac{\sin\omega_P n}{\pi n},\quad -\infty<n<+\infty$$

- NOTE 1: these are zero-phase filters !
- NOTE 2: these filters are computationally unrealizable, why ?

Basic concepts

- Phase distortion and group delay

$\rightarrow$ usually it is desired that a system exhibits linear phase response since this denotes a simple system delay:

- Example 1: $h[n]=\delta[n-n_d] \overset{F}{\longleftrightarrow} H(e^{j\omega})=e^{-j\omega n_d} \quad \therefore \quad \angle H(e^{j\omega})=-\omega n_d$

- Example 2:

$$H_{PB}(e^{j\omega})=\begin{cases} e^{-j\omega n_d}, & |\omega|<\omega_p \\ 0, & \omega_p<|\omega|\le\pi \end{cases} \overset{F}{\longleftrightarrow} h_{PB}[n]=\frac{\sin\omega_P(n-n_d)}{\pi(n-n_d)} \quad , \quad -\infty<n<+\infty$$

$\rightarrow$ deviations to the linear phase response are known as phase distortions and are better characterized using the concept of group delay:

$$\tau(\omega)=-\frac{d}{d\omega}\{\angle H(e^{j\omega})\}$$

$\rightarrow$ the group delay gives the delay of the system as a function of frequency. It is clear that in the case of linear phase systems, the group delay is <u>constant</u>.

Basic concepts

- The difference equation and the transfer function

→ another alternative allowing to describe a discrete-time system is the constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell] \quad \Leftrightarrow \quad y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{\ell=0}^{M} \frac{b_\ell}{a_0} x[n-\ell]$$

→ if the initial conditions are complete rest, the discrete-time system is causal, linear and time-invariant. Under these circumstances, the characteristics and properties of the LTI system are better studied using the Z transform:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell] \quad \overset{Z}{\longleftrightarrow} \quad \sum_{k=0}^{N} a_k Z^{-k} Y(z) = \sum_{\ell=0}^{M} b_\ell Z^{-\ell} X(z)$$

from which we derive the algebraic form of the transfer function of the system:

$$H(z) \overset{\Delta}{=} \frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{M} b_\ell Z^{-\ell}}{\sum_{k=0}^{N} a_k Z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{\ell=1}^{M} \left(1 - c_\ell Z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - d_k Z^{-1}\right)}$$

– Example: find the difference equation of the LTI system characterized by:

$$H(z)=\frac{\left(1+Z^{-1}\right)^2}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1+\frac{3}{4}Z^{-1}\right)}$$

Solution: H(z) may be presented as:

$$H(z)=\frac{1+2Z^{-1}+Z^{-2}}{1+\frac{1}{4}Z^{-1}-\frac{3}{8}Z^{-2}}=\frac{Y(z)}{X(z)}$$

from where we conclude easily that:

$$y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2]=x[n]+2x[n-1]+x[n-2]$$

- **Question**: what is the importance, in this problem, of specifying the RC of H(z) ?
  another way to formulate the question: is the difference equation a complete form characterizing an LTI system ?

- Concept of stability and causality

→ as already seen, these are not mutually dependent concepts

- Inverse system
  - → given a system function H(z), its inverse is the system function $H_i(z)$ such that if it is cascaded with H(z), the resulting system is an all-pass system:

$$G(z)=H(z)\cdot H_i(z)=1 \quad \Rightarrow \quad H_i(z)=\frac{1}{H(z)} \quad \overset{Z}{\longleftrightarrow} \quad g[n]=h[n]*h_i[n]=\delta[n]$$

  - → it results that the frequency response of the inverse system, if it exists, it is given by:

$$H_i\left(e^{j\omega}\right)=\frac{1}{H\left(e^{j\omega}\right)}$$

    - **NOTE**: not all systems have an inverse system (*i.e.* a reciprocal) !

  - → in particular, rational system functions have a reciprocal that is easy to identify:

$$H(z)=\frac{b_0}{a_0}\frac{\prod_{\ell=1}^{M}\left(1-c_\ell Z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k Z^{-1}\right)} \longrightarrow H_i(z)=\frac{a_0}{b_0}\frac{\prod_{k=1}^{N}\left(1-d_k Z^{-1}\right)}{\prod_{\ell=1}^{M}\left(1-c_\ell Z^{-1}\right)}$$

where the poles of H(z) are the zeroes of $H_i(z)$ and vice-versa.

Basic concepts

Question: what is the region of convergence of $H_i(z)$? Is it unique ?

A: It may be any RC (*i.e.* it is valid) as long as it overlaps with that of H(z).

Example:

$$H(z)=\frac{1-0.5Z^{-1}}{1-0.9Z^{-1}} \quad , \quad |z|>0.9 \longrightarrow H_i(z)=\frac{1-0.9Z^{-1}}{1-0.5Z^{-1}} \quad , \quad RC=? \quad , \quad \begin{cases}|z|>0.5 & ? \\ |z|<0.5 & ?\end{cases}$$

it is only valid the one that overlaps with $|z|>0.9 \;\therefore\; |z|>0.5$

and thus: $h[n]=0.9^n u[n]-0.5(0.9)^{n-1}u[n-1]$, (causal and stable)

$h_i[n]=0.5^n u[n]-0.9(0.5)^{n-1}u[n-1]$, (causal and stable)

Important particular case: a stable and causal LTI system has an inverse that is also stable and causal if <u>all of its zeroes and poles are inside the unit circumference</u>, *i.e.* if it is a minimum-phase system (topic to be developed later on …)

Example: given the 2nd order causal system:

$$H(z) = \frac{1}{\left(1 - re^{j\theta}Z^{-1}\right)\left(1 - re^{-j\theta}Z^{-1}\right)} \quad , \quad |z| > r$$

◊ find its difference equation, its impulse response and represent its frequency response magnitude, phase and group delay characteristics (consider r=0.8 and $\theta=\pi/3$). Repeat the problem for the inverse system.

A: we may write: $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta\, Z^{-1} + r^2 Z^{-2}}$$

from where we conclude: $y[n]=x[n]+2r\cdot\cos\theta\cdot y[n-1]-r^2y[n-2]$

concerning its impulse response, we also have:

$$H(z) = \frac{1}{\left(1 - re^{j\theta}Z^{-1}\right)\left(1 - re^{-j\theta}Z^{-1}\right)} =$$

$$= \frac{1/\left(1 - e^{-j2\theta}\right)}{1 - re^{j\theta}Z^{-1}} + \frac{1/\left(1 - e^{j2\theta}\right)}{1 - re^{-j\theta}Z^{-1}} \quad , \quad |z| > r \quad \overset{Z}{\longleftrightarrow} \quad h[n] = \frac{r^n \sin\left(n+1\right)\theta}{\sin\theta} u[n]$$

Regarding its frequency response, we start from: $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \dfrac{1}{\left(1-re^{j(\theta-\omega)}\right)\left(1-re^{-j(\theta+\omega)}\right)}$

Using the logarithmic scale for the magnitude response (which is common), we get:

$$20\log\left|H(e^{j\omega})\right| = 10\log\left|H(e^{j\omega})\right|^2 = 10\log\left[H(e^{j\omega})\cdot H^*(e^{j\omega})\right]$$

and finding first $H(e^{j\omega})\cdot H^*(e^{j\omega})$:

- **NOTE**: it is easy to verify that: $\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})\cdot H^*(e^{j\omega}) = H(z)\cdot H^*(1/z^*)\big|_{Z=e^{j\omega}}$

hint: find the convolution between h[n] and $h^*[-n]$

$$H(e^{j\omega})\cdot H^*(e^{j\omega}) = \frac{1}{1-2r\cos(\theta-\omega)+r^2}\cdot\frac{1}{1-2r\cos(\theta+\omega)+r^2}$$

we have then: $20\log\left|H(e^{j\omega})\right| = -10\log\left[1-2r\cos(\theta-\omega)+r^2\right] - 10\log\left[1-2r\cos(\theta+\omega)+r^2\right]$

As we conclude from the initial expression, phase is given by:

$$\angle H(e^{j\omega}) = \arctan\frac{r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)} - \arctan\frac{r\sin(\theta+\omega)}{1-r\cos(\theta+\omega)}$$

and the group delay, recalling in the meantime that: $\dfrac{d}{d\omega}\{\arctan f(\omega)\} = \dfrac{\frac{d}{d\omega}\{f(\omega)\}}{1+[f(\omega)]^2}$

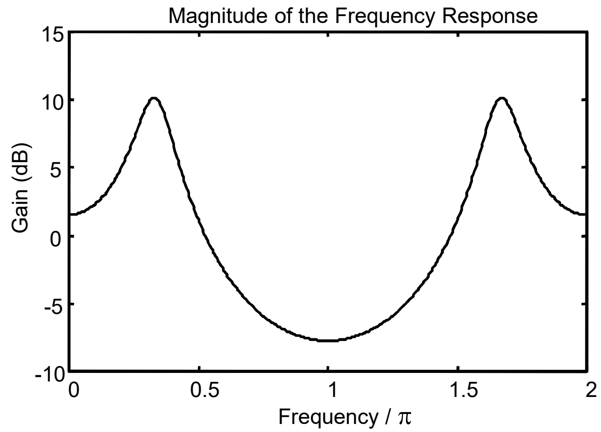
is given by: $\tau(\omega) = -\dfrac{d}{d\omega}\{\angle H(e^{j\omega})\} = -\dfrac{r^2-r\cos(\theta-\omega)}{1-2r\cos(\theta-\omega)+r^2} - \dfrac{r^2-r\cos(\theta+\omega)}{1-2r\cos(\theta+\omega)+r^2}$

Basic concepts

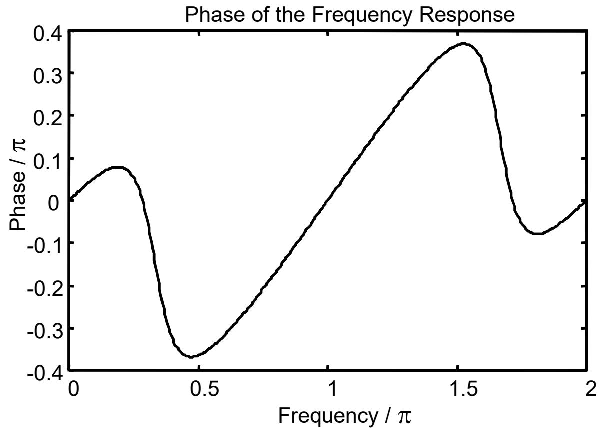
The representation presumes r=0.8 and $\theta=\pi/3$. The three figures have been obtained using the following Matlab code:

```
teta=pi/3.0; erre=0.8; N=512;
b=1.0; a=[1 -2*erre*cos(teta) erre.^2];
[H,W] = FREQZ(b,a,N,'whole');
figure(1)
plot(W/pi,20*log10(abs(H)));
ylabel('Gain (dB)');
xlabel(‘Frequency / \pi');
title(‘Magnitude of the Frequency Response');
figure(2)
plot(W/pi,angle(H)/pi);
ylabel(‘Phase / \pi');
xlabel('Frequency / \pi');
title(‘Phase of the Frequency Response');
figure(3)
plot(W/pi,grpdelay(b,a,N,'whole'));
ylabel(‘Group Delay in Samples');
xlabel('Frequency / \pi');
title(‘Group Delay');
```

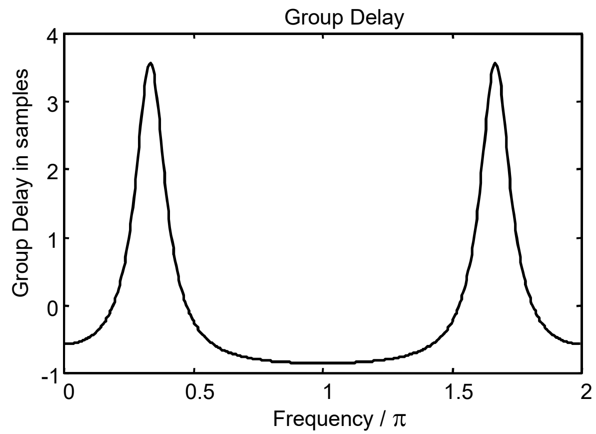
NOTE: although it was not necessary in this example, it is usual to wrap the phase representation considering the fundamental period $[-\pi, \pi[$



$|H(e^{j\omega})|$



$\angle H(e^{j\omega})$



$\tau(\omega)$

Basic concepts

The inverse system is:

$$H_i(z) = 1 - 2r\cos\theta\, Z^{-1} + r^2 Z^{-2} \quad , \quad |Z| > 0 \quad \overset{Z}{\longleftrightarrow} \quad h[n] = \delta[n] - 2r\cos\theta\,\delta[n-1] + r^2\delta[n-2]$$

whose frequency response is given by:

$$H_i(e^{j\omega}) = H_i(z)\Big|_{z=e^{j\omega}} = \left(1 - re^{j(\theta-\omega)}\right)\left(1 - re^{-j(\theta+\omega)}\right)$$

since it is the reciprocal of the system seen previously, its log magnitude is given by:

$$20\log\left|H_i(e^{j\omega})\right| = 20\log\left|\frac{1}{H(e^{j\omega})}\right| = -20\log\left|H(e^{j\omega})\right|$$

thus, we conclude that the dB gain of the inverse function corresponds to the attenuation (=negative gain) of function $H(e^{j\omega})$. Therefore, the plot of the gain of the inverse function corresponds to the symmetric (on the log domain) of that of the initial system (*i.e.* the ordinate scale is just made symmetric).

We conclude easily that the same happens regarding the phase and group delay representations (in Matlab it is sufficient to switch vectors 'a' and 'b').

All-pass systems

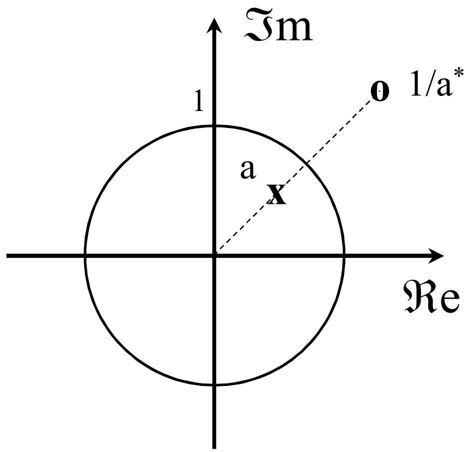
- Definition

→ are systems whose frequency response magnitude does not depend on ω. The basic Z function of a first-order all-pass system may be presented as:

$$H_{AP}(z) = \frac{Z^{-1} - a^*}{1 - aZ^{-1}}$$

It can be verified that $|H_{PT}(e^{j\omega})|=1$: $H_{AP}(z) \cdot H^*_{AP}(1/z^*) = \frac{Z^{-1} - a^*}{1 - aZ^{-1}} \cdot \frac{Z - a}{1 - a^* Z} = \frac{1 + |a|^2 - a^* Z - aZ^{-1}}{1 + |a|^2 - a^* Z - aZ^{-1}} = 1$

→ the main feature of a first-order all-pass system is that once the pole position is known, the position of the zero is at the reciprocal-conjugate of that of the pole:



NOTE: we admit here $|a|<1$

→ in general, an all-pass system may aggregate an arbitrary number of first-order all-pass functions.

→ all-pass systems have various applications such as to compensate phase distortions introduced by other systems.

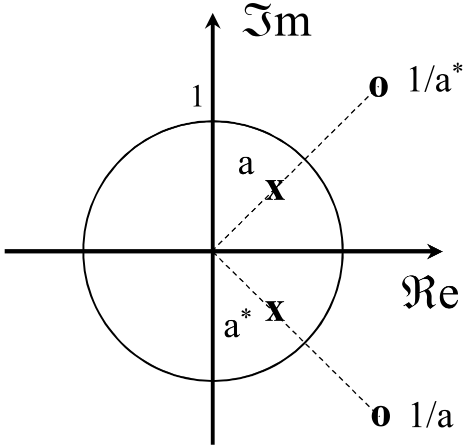
- Example
  - if an all-pass system has a real-valued impulse response, its poles and zeroes are either real or occur in complex conjugate pairs; *e.g.* (admitting causality):

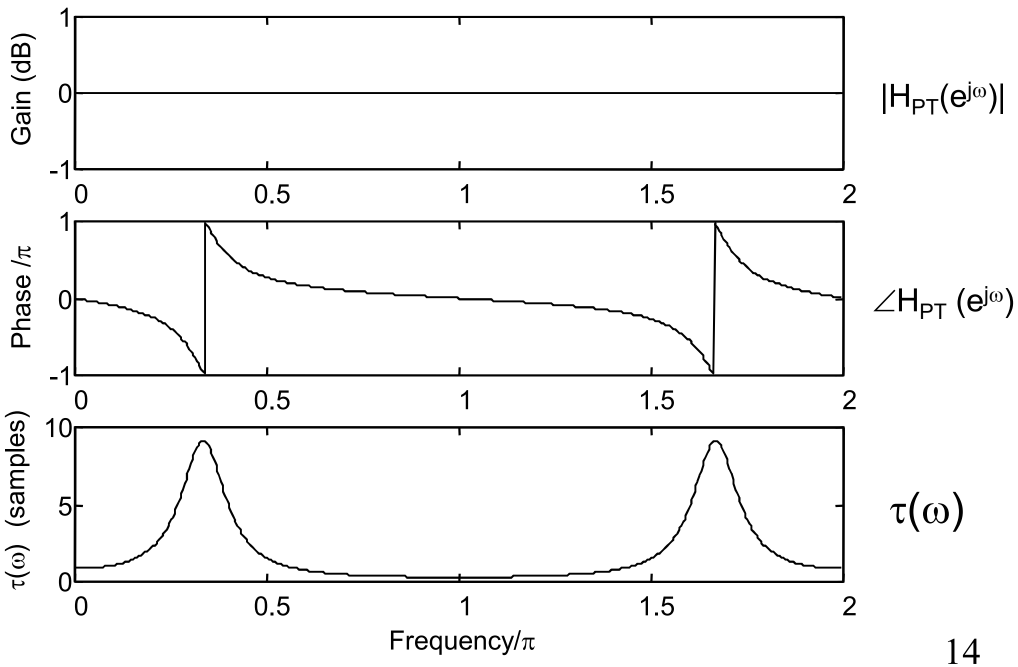
$$H_{PT}(z) = \frac{Z^{-1} - a^*}{1 - aZ^{-1}} \cdot \frac{Z^{-1} - a}{1 - a^* Z^{-1}} = \frac{|a|^2 - (a + a^*)Z^{-1} + Z^{-2}}{1 - (a + a^*)Z^{-1} + |a|^2 Z^{-2}} \quad , \quad |Z| > |a|$$

taking, for example $a = 0.8e^{j\pi/3}$, the transfer function results:

$$H_{PT}(z) = \frac{0.64 - 0.8Z^{-1} + Z^{-2}}{1 - 0.8Z^{-1} + 0.64Z^{-2}} \quad , \quad |Z| > 0.8$$

which gives rise to the represented diagrams.





Minimum-phase and maximum phase systems

- Definition

→ a stable and causal discrete-time system, H(z), has all of its poles inside the unit circumference

  - this statement does not constrain the zeroes

→ in certain cases, it is useful to request that the inverse system $H_i(z)=1/H(z)$ is also stable and causal

  - as the poles of $H_i(z)$ are the zeroes of H(z), this request implies that H(z) has all of its zeros *and* poles inside the unit circumference

→ a minimum-phase system complies with these conditions, *i.e.* either its transfer function or that if its inverse, are causal and stable

→ a causal and stable discrete-time system that is characterized by a rational transfer function may always be expressed as: $H(z) = H_{\min}(z) \cdot H_{PT}(z)$ where $H_{min}(z)$ represents a minimum phase system, and $H_{PT}(z)$ represents one or more first-order all-pass systems.

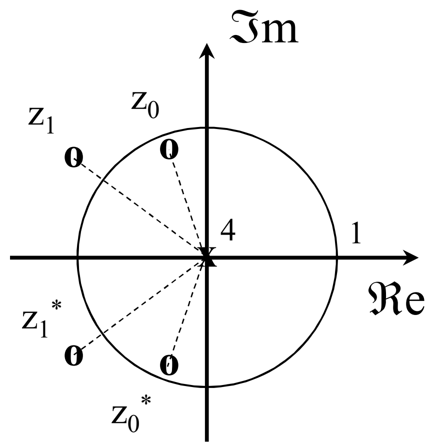
Minimum-phase and maximum phase systems

→ the previous statement also means that any causal and stable system may be converted into a minimum phase system *without* any modification to its frequency response magnitude, by "reflecting" the zeroes outside the unit circumference to the reciprocal conjugate positions (inside the unit circle), using first-order all-pass filters:

$$H_{\min}(z) = H(z) \cdot H_{PT}(z)$$

- Example
  - considering a zero-pole diagram including four poles at z=0 and four zeroes at $Z_0=0.9e^{j0.6\pi}$, Z_0^*, $Z_1=1.25e^{j0.8\pi}$, Z_1^*, find all stable and causal systems having real impulse response and showing the same frequency response magnitude, identify also which one has a stable inverse

A: the first system is: $H_1(z) = \dfrac{(Z-Z_0)(Z-Z_0^*)(Z-Z_1)(Z-Z_1^*)}{Z^4}$

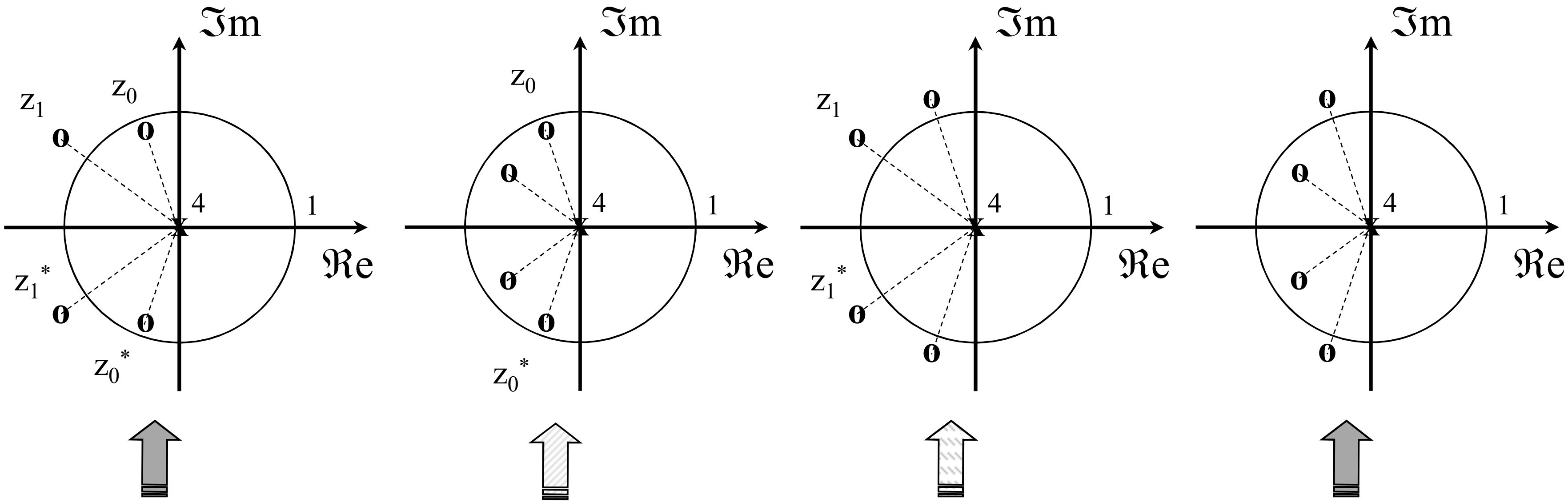


Considering the distribution of poles and zeros we conclude that all systems are FIR and that, in order to be causal, only zero or negative powers of Z may exist in the transfer function.

Minimum-phase and maximum phase systems

- Example (*cont.*)

  – we also conclude that considering the real impulse response requirement, three other systems exist that have the same frequency response magnitude as $H_1(z)$, given that their transfer function results from $H_1(z)$ by "reflecting" (in the sense of reciprocal conjugate) zeroes (in fact pairs of zeroes, why ?), either to the inside, or to the outside of the unit circumference.



$H_1(z)$ $\quad$ $H_{\min}(z) = H_1(z)H_{PT1}(z)$ $\quad$ $H_{\max}(z) = H_1(z)H_{PT0}(z)$ $\quad$ $H_2(z) = H_1(z)H_{PT0}(z)H_{PT1}(z)$

where: $H_{PT0}(z) = \frac{Z^{-1}-a^*}{1-aZ^{-1}} \cdot \frac{Z^{-1}-a}{1-a^*Z^{-1}} \quad , \quad a = Z_0$ $\qquad$ $H_{PT1}(z) = \frac{Z^{-1}-a^*}{1-aZ^{-1}} \cdot \frac{Z^{-1}-a}{1-a^*Z^{-1}} \quad , \quad a = Z_1$

Minimum-phase and maximum phase systems

- Example (*cont.*)

  – the minimum phase system $H_{min}(z)$ has transfer function:

$$H_{\min}(z) = |Z_1|^2 \frac{(Z - Z_0)(Z - Z_0^*)(Z - 1/Z_1)(Z - 1/Z_1^*)}{Z^4}$$

this is the only system whose inverse is also stable.

  – the maximum phase system $H_{max}(z)$ has transfer function:

$$H_{\max}(z) = |Z_0|^2 \frac{(Z - 1/Z_0)(Z - 1/Z_0^*)(Z - Z_1)(Z - Z_1^*)}{Z^4}$$

  – and, finally, the other possible alternative $H_2(z)$ has transfer function:

$$H_2(z) = |Z_0|^2 |Z_1|^2 \frac{(Z - 1/Z_0)(Z - 1/Z_0^*)(Z - 1/Z_1)(Z - 1/Z_1^*)}{Z^4}$$

NOTE: we will see later that in the case of FIR systems, as in this example, there is a precise relation between the minimum-phase system and the maximum-phase system.

Minimum-phase and maximum phase systems

- Properties of minimum-phase systems
  - since a non minimum-phase system is obtained from a minimum-phase system by adding all-pass filters whose phase is always negative for $0<\omega<\pi$, when zeroes are “reflected” from the inside to the outside of the unit circumference, we conclude that among all systems having the same frequency response magnitude, the minimum-phase system shows a “less negative” phase,
  - from the previous it results that the group delay is minimized in the case of a minimum phase system,
  - on the other hand, it also results from the previous property that the energy of the impulse response of a causal minimum-phase system is more concentrated near n=0 (*i.e.* minimum energy delay)

QUESTION: how are these properties adapted to the case of maximum-phase systems ?

NOTE: the concept of minimum-phase and maximum-phase is valid either for FIR or IIR systems

Linear-phase systems

- Definition
  - systems whose frequency response has the form:

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha} \quad , \quad |\omega| < \pi$$

where α is a quantity (integer or non-integer) denoting the constant group delay of the system (*i.e.* all frequencies are equally delayed), in other words, the phase of $H(e^{j\omega})$ is a linear function of ω:

$$\angle H(e^{j\omega}) = -\omega\alpha$$

However, in general terms, we define generalized linear-phase system, a system which may be expressed as:

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j(\omega\alpha - \beta)} \quad , \quad |\omega| < \pi$$

where $A(e^{j\omega})$ is a real function (it may be positive or negative) and α and β are constants making that

$$\angle H(e^{j\omega}) = \beta - \omega\alpha$$

is a linear function of ω.

Linear-phase systems

Thus, independently of the sudden "jumps" the phase function may exhibit, the group delay is always a constant:

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\} = \alpha$$

In the following, we will consider that 2α is always an integer. In this case, it can be shown [Oppenheim, section 5.7.1-5.7.2] that the requirement of generalized linear-phase implies that a real-valued h[n] has a form of symmetry around α:

- even symmetry: $h[2\alpha-n] = h[n]$
- odd symmetry: $h[2\alpha-n] = -h[n]$

In order to simplify the discussion, in the following we presume causality which, when combined with the fact that α is finite, implies that the length of the impulse response is also finite (*i.e.* the system is FIR). Considering that the length may be even of odd and that symmetry may also be even or odd, it results that there are four types of linear phase systems (or filters).

Linear-phase systems

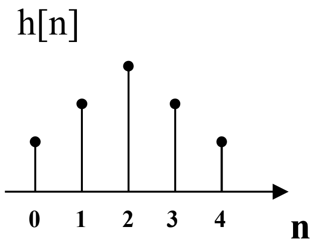
$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n} = \underbrace{A(e^{j\omega})}_{\text{real-valued}} e^{-j(\omega\alpha-\beta)}$$

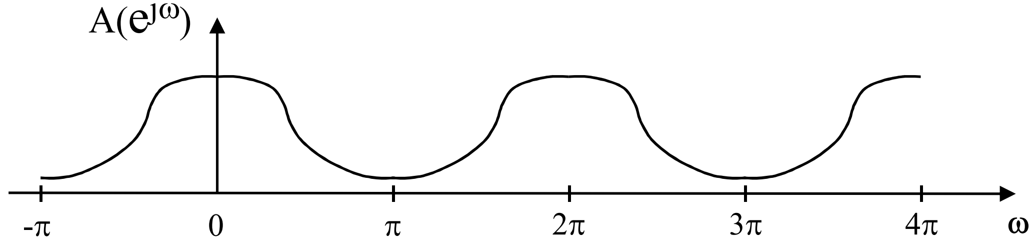
- Type 1 FIR linear-phase systems

→ N is odd

→ h[n] is symmetric ∴ h[n] = h[N-1-n]

example: N=5





→ particularities:

- $\beta=0$ or $\beta=\pi$
- $\alpha=(N-1)/2$ (integer)
- $A(e^{j\omega})$ is even around $\omega=0$ ∴ $A(e^{j\omega}) = A(e^{-j\omega})$
- $A(e^{j\omega})$ is even around $\omega=\pi$ ∴ $A(e^{j(\pi+\omega)}) = A(e^{j(\pi-\omega)})$
- $A(e^{j\omega})$ is periodic with period 2π ∴ $A(e^{j(\omega+2\pi)}) = A(e^{j\omega})$

Linear-phase systems

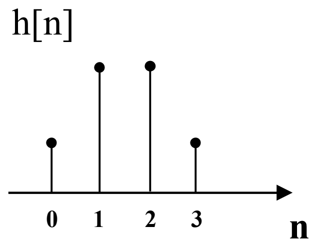
$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n} = \underbrace{A(e^{j\omega})}_{\text{real-valued}} e^{-j(\omega\alpha - \beta)}$$

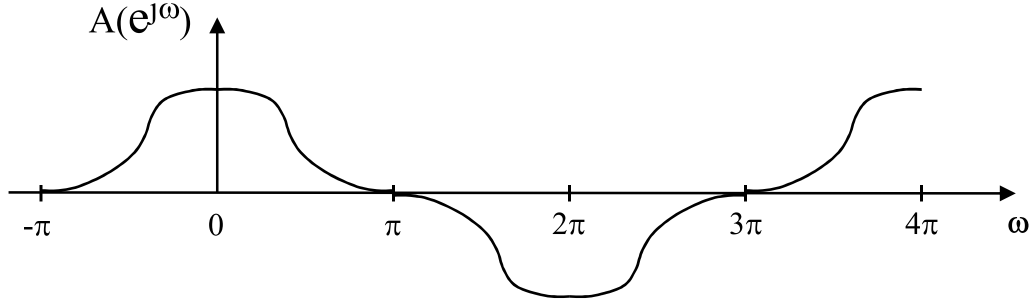
- Type 2 FIR linear-phase systems

→ N is even

→ h[n] is symmetric ∴ $h[n] = h[N-1-n]$

example: N=4





→ particularities:

- $\beta=0$ or $\beta=\pi$
- $\alpha=(N-1)/2$ (integer +1/2)
- $A(e^{j\omega})$ is even around $\omega=0$ ∴ $A(e^{j\omega}) = A(e^{-j\omega})$
- $A(e^{j\omega})$ is odd around $\omega=\pi$ ∴ $A(e^{j(\pi+\omega)}) = -A(e^{j(\pi-\omega)})$
- $A(e^{j\omega})$ is periodic with period 4π ∴ $A(e^{j(\omega+4\pi)}) = A(e^{j\omega})$

Linear-phase systems

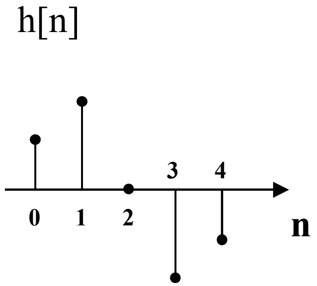
$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n} = \underbrace{A(e^{j\omega})}_{\text{real-valued}} e^{-j(\omega\alpha - \beta)}$$

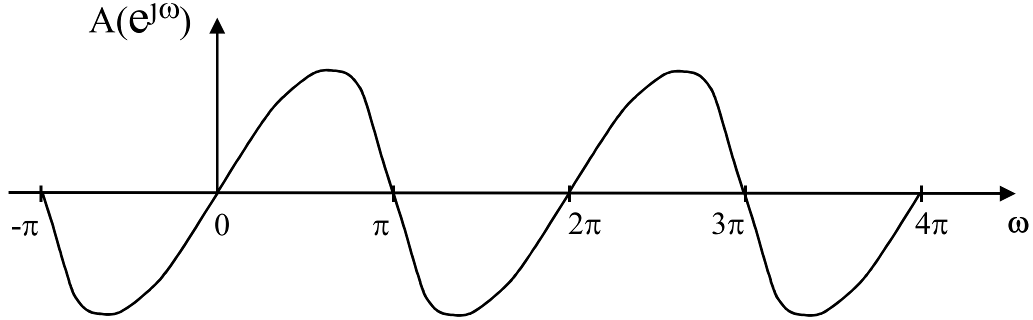
- Type 3 FIR linear-phase systems

→ N is odd

→ h[n] is anti-symmetric ∴ $h[n] = -h[N-1-n]$

example: N=5





→ particularities:

- $\beta=\pi/2$ or $\beta=3\pi/2$
- $\alpha=(N-1)/2$ (integer)
- $A(e^{j\omega})$ is odd around $\omega=0$ ∴ $A(e^{j\omega}) = -A(e^{-j\omega})$
- $A(e^{j\omega})$ is odd around $\omega=\pi$ ∴ $A(e^{j(\pi+\omega)}) = -A(e^{j(\pi-\omega)})$
- $A(e^{j\omega})$ is periodic with period 2π ∴ $A(e^{j(\omega+2\pi)}) = A(e^{j\omega})$

Linear-phase systems

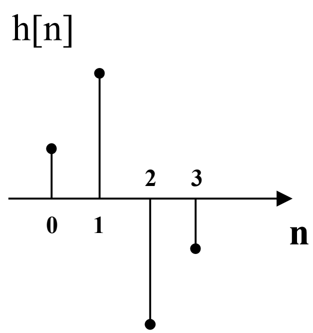
$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n} = \underbrace{A(e^{j\omega})}_{\text{real-valued}} e^{-j(\omega\alpha-\beta)}$$

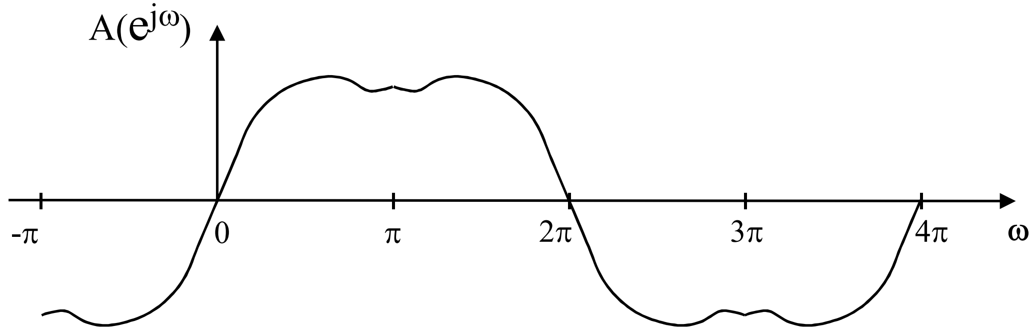
- Type 4 FIR linear-phase systems

  $\rightarrow$ N is even

  $\rightarrow$ h[n] is anti-symmetric $\therefore$ h[n] = -h[N-1-n]

example: N=4





$\rightarrow$ particularities:

- $\beta = \pi/2$ or $\beta = 3\pi/2$
- $\alpha = (N-1)/2$ (integer +1/2)
- $A(e^{j\omega})$ is odd around $\omega=0$ $\therefore A(e^{j\omega}) = -A(e^{-j\omega})$
- $A(e^{j\omega})$ is even around $\omega=\pi$ $\therefore A(e^{j(\pi+\omega)}) = A(e^{j(\pi-\omega)})$
- $A(e^{j\omega})$ is periodic with period 4π $\therefore A(e^{j(\omega+4\pi)}) = A(e^{j\omega})$

Linear-phase systems

- summary

from the previous it results that:

$\rightarrow$ types 3 and 4 are not suitable to realize low-pass filters

- because $A(e^{j0})=0$ (i.e. a zero exists at Z=1)

$\rightarrow$ types 2 and 3 are not suitable to realize high-pass filters

- because $A(e^{j\pi})=0$ (i.e. a zero exists at Z=-1)

$\rightarrow$ types 3 and 4 give rise to a constant phase shift ($\pi/2$ or $-\pi/2$)

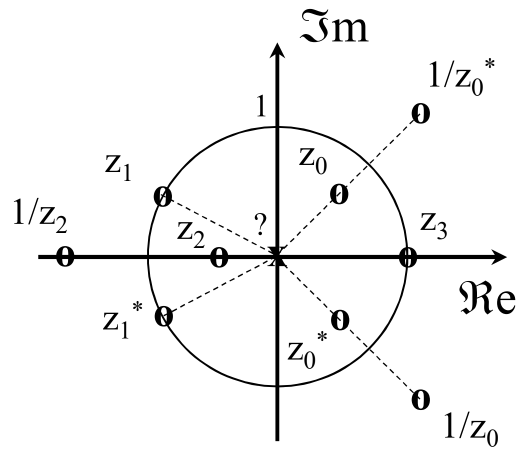
- which is desirable in the case of differentiators or Hilbert transformers

- Locations of zeroes for FIR linear-phase filters
  - It can be shown [Oppenheim, section 5.7.3] that if $Z_0=re^{j\theta}$ is a zero of H(z), where H(z) is a linear-phase system having a real-valued impulse response, then Z_0 belongs to a set of 4 zeroes (having reciprocal conjugate relations): Z_0 , Z_0^* , 1/ Z_0 , e 1/ Z_0^*

→ particular cases:

- if Z_0 is on the unit circumference, then the group is limited to 2 zeroes
- if Z_0 is real, then the group is limited to 2 zeroes
- if Z_0 is real and is on the unit circumference, then the group is limited to a single zero

Example: distribution of the zeroes pertaining to a linear-phase FIR system



Question 1: what is the multiplicity of the pole at the origin ?

Question 2: what is the type of this FIR system ?

Linear-phase systems

- Relation between linear, minimum and maximum phase FIR systems

  – If $H_{min}(z)$ is an FIR minimum-phase system, the corresponding maximum phase system is readily obtained using:

$$H_{max}(z)=Z^{-L}\,H_{min}(z^{-1}),$$ where L is the order of H(z)

$\rightarrow$ Question 1: what is the interpretation of this relation in the n domain, i.e. considering $h_{min}[n]$ and $h_{max}[n]$?

$\rightarrow$ Question 2: what is the result of the following Z product:

$$H(z)=H_{min}(z)\cdot H_{max}(z)$$

?